Number of Knots and Links in Linear K_7

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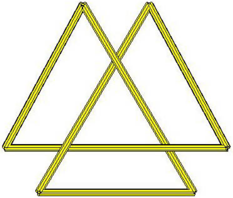
Contents

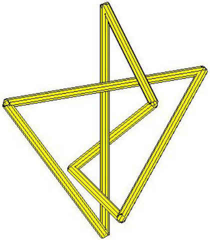
Knots and Links

- A m-component *link* is a union of m disjoint circles embedded in $\mathbb{R}^3$.
- A *knot* is a 1-component link.
- Two links L and L' are *ambient isotopic*, $L \sim L'$, if there exists a self-homeomorphism Φ on $\mathbb{R}^3$ such that $\Phi(L) = L'$.
- The *link type* of L is the ambient isotopy class of L.
- L is *trivial* if $L \sim$ disjoint circles embedded in a plane of $\mathbb{R}^3$.

Polygonal Links (1)

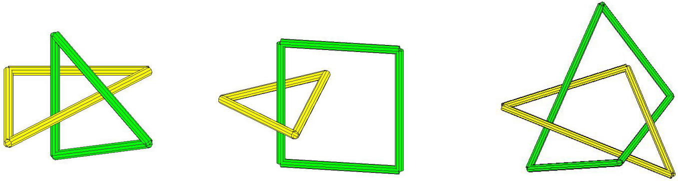
- A *polygonal link* is a link consisting of finitely many straight line segments(*edges*) whose endpoints are called *vertices*.
- Trefoil knot, Figure-8 knot





Polygonal Links (2)

- Hopf link, $\{2,4\}$-Torus link



- For a link type $\mathcal{L}$, *polygon index $p(\mathcal{L})$* is the minimal number of the line segments required to realize $\mathcal{L}$ as a polygon or a union of polygons.
- $p(\mathit{Trefoil}) = 6$, $p(\mathit{Figure}\text{-}8) = 7$, $p(\mathit{Hopf\ link}) = 6$, $p(\{2,4\}\text{-}\mathit{Tours\ link}) = 7$ and $p(\mathcal{L}) \geq 8$ for any other non-trivial link type $\mathcal{L}$.

Complete Graphs and Linear Embeddings

- K_n is the *complete graph* with n vertices.
- A *linear (embedding of) K_n* is an embedding of K_n into $\mathbb{R}^3$ such that each edge of K_n is mapped onto a line segment.
- A subgraph of K_n which is homeomorphic to the circle is called a *cycle* of K_n.
- A cycle of K_n is a *k-cycle* if it contains exactly k edges. An n-cycle of K_n is called a *Hamiltonian* cycle.
- *(k, l)-cycle* is a disjoint pair of a k-cycle and an l-cycle of K_n.

Knots and Links in K_n

Theorem (Conway-Gordon (1983))

- *Any embedding of K_6 contains at least one non-trivial link as one of its $(3,3)$-cycle.*
- *Any embedding of K_7 contains at least one non-trivial knot as one of its Hamiltonian cycle.*

Theorem (Negami (1991))

For any knot type $\mathcal{K}$, there exists a natural number $N(\mathcal{K})$ such that every linear embedding of $K_{N(\mathcal{K})}$ contains a polygonal knot of the type $\mathcal{K}$.

Links in Linear K_7

Theorem (Hughes (2006))

A linear K_6 contains at most three Hopf links.

Theorem (Fleming-Mellor (2009))

Any linear K_7 contains at least twenty-one non-trivial links.

Theorem (Ludwig-Arbisi (2009))

The minimum number of non-trivial links in any linear K_7 which forms a convex polyhedron of seven vertices is twenty-one and the maximum number of non-trivial links in K_7 is forty-eight.

Knots in Linear K_7

Theorem (Brown (1977), Alfonsín (1999), Nikkuni (2008))

Any linear K_7 contains at least one trefoil knot. $N(trefoil) = 7$.

Theorem (Foisy-Ludwig (2009))

A linear embedding of K_7 which forms a convex polyhedron of seven vertices may have at most fourteen non-trivial knots.

Theorem (Huh (2009))

A linear K_7 contains at most three figure-8 knots.

Knots and Links in Linear K_7 (1)

Theorem (Huh-Jeon (2007), Nikkuni (2008))

A linear K_6 contains at most one trefoil and at most three Hopf links.

- *A linear K_6 contains a trefoil if and only if it contains exactly three Hopf links.*
- *A linear K_6 does not contain a trefoil if and only if it contains exactly one Hopf link.*

Number of Knots and Links in Linear K_6	
Hopf	Trefoil
1	0
3	1

Knots and Links in Linear K_7 (2)

Corollary (Nikkuni (2008))

A linear K_7 contains n trefoils as its 6-cycles if and only if it contains $2n+7$ Hopf links as its $(3,3)$-cycles.

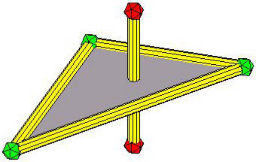
Theorem (Nikkuni (2008))

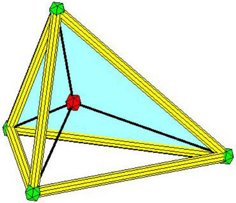
For any linear embedding f of K_7, $\sum_{\gamma\in\Gamma_7(K_7)} a_2(f(\gamma)) = 1$ if and only if non-trivial 2-component links in $f(K_7)$ are exactly twenty-one Hopf links.

Corollary

A linear K_7 contains one more trefoils than figure-8 knots among its 7-cycles if and only if it contains exactly twenty-one Hopf links.

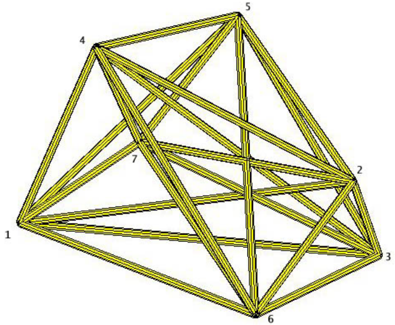
5 Points in $\mathbb{R}^3$, Signed Circuits





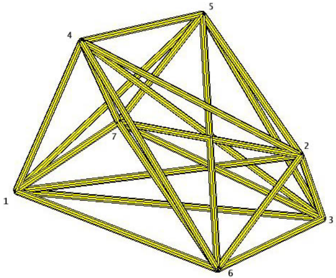
- $C = \{1, 2, 3, -4, -5\}$ and $-C = \{-1, -2, -3, 4, 5\}$ where $C^+ = \{1, 2, 3\}$, $C^- = \{4, 5\}$
- $C = \{1, 2, 3, 4, -5\}$ and $-C = \{-1, -2, -3, -4, 5\}$ where $C^+ = \{1, 2, 3, 4\}$, $C^- = \{5\}$
- (Radon's Theorem) Any set of $d + 2$ points in $\mathbb{R}^d$ can be partitioned into two disjoint sets whose convex hulls intersect.

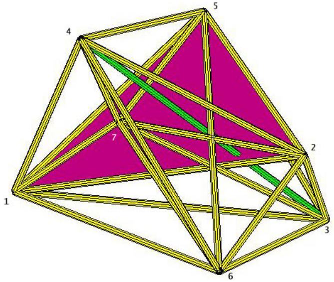
7 Vertices of a Linear K_7 in $\mathbb{R}^3$



$1 = [9.239689, 0.635179, 3.144147]$, $2 = [2.429298, 7.173933, 3.78047]$, $3 = [2.028693, 7.780807, 0.9256]$, $4 = [6.725992, 1.339026, 8.642522]$, $5 = [0.8746, 1.629319, 5.610789]$, $6 = [8.836616, 8.955078, 4.544055]$, $7 = [7.11866, 3.255548, 6.418356]$

Oriented Matroid of a Linear K_7





$$\begin{aligned}\mathcal{C} = \{&\{1,2,-3,-4,5\}, \{1,2,-3,-4,-6\}, \{1,2,-3,-4,-7\}, \{1,2,-3,-5,-6\},\\ &\{1,2,-3,-5,-7\}, \{1,2,-3,6,-7\}, \{1,2,4,-5,-6\}, \{1,2,4,-5,-7\},\\ &\{1,2,4,6,-7\}, \{1,2,-5,-6,7\}, \{1,-3,-4,5,6\}, \{1,3,4,-5,-7\},\\ &\{1,3,4,6,-7\}, \{1,-3,5,6,-7\}, \{1,4,5,6,-7\}, \{2,3,4,-5,-6\},\\ &\{2,-3,-4,5,7\}, \{2,-3,-4,-6,7\}, \{2,3,-5,-6,7\}, \{2,-4,-5,-6,7\},\\ &\{3,4,-5,-6,-7\}\}\end{aligned}$$

Oriented Matroids

- An *oriented matroid* $\mathcal{M}$ on a finite set E is defined by its collection $\mathcal{C}$ of *signed curcuits* satisfying the following three properties;
  - (a) For all $C \in \mathcal{C}$, $C \neq \emptyset$
  - (b) For all $C_1, C_2 \in \mathcal{C}$, $C_2 \subseteq C_1$ implies $C_2 = C_1$ or $C_2 = -C_1$
  - (c) (Elimination property) For all $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq -C_2$ and all $x \in (C_1^+ \cap C_2^-)$, there exists $C_3 \in \mathcal{C}$ such that $C_3^+ \subseteq (C_1^+ \cup C_2^+) \setminus \{x\}$ and $C_3^- \subseteq (C_1^- \cup C_2^-) \setminus \{x\}$.
- An oriented matroid $\mathcal{M}$ is *acyclic* if it has no circuit C with $C^- = \emptyset$.
- Matroids obtained from a linear K_7 in $\mathbb{R}^3$ are called *uniform affine acyclic oriented matroids* of rank $4(= 3 + 1)$.

Knots and Links from Oriented Matroids (1)

Theorem (Jeon-Jin)

Uniform affine acyclic oriented matroids of rank 4 on 7 vertices determine the number of knots and links of all linear K_7.

(Idea of Proof) Let f be a linear embedding of K_7 and $\mathcal{M}(f)$ be the oriented matroid of f. Then we will see that the number of links in f is determined by $\mathcal{M}(f)$. Note that the number of 6-trefoils in f is determined by the number of $(3,3)$-Hopf links. Hence number of 6-trefoils in f is determined by $\mathcal{M}(f)$.
By Nikkuni(2008), the following holds.

$$7\sum_{\gamma\in\Gamma_7(K_7)} a_2(f(\gamma)) - 6\sum_{\gamma\in\Gamma_6(K_7)} a_2(f(\gamma)) = 2\sum_{\gamma\in\Gamma_{4,3}(K_7)} \mathrm{lk}(f(\gamma))^2 - 21$$

Knots and Links from Oriented Matroids (2)

Hence $\sum_{\gamma\in\Gamma_7(K_7)} a_2(f(\gamma))$ is determined by $\mathcal{M}(f)$. Note that $a_2(f(\mathit{trefoil})) = 1$ and $a_2(f(\mathit{figure}-8)) = -1$.
By Huh(2009), we can see that the number of figure-8 knots in f is determined by $\mathcal{M}(f)$. Therefore the number of 7-trefoils is also determined by $\mathcal{M}(f)$.

Corollary

Two linear embeddings of K_7 which realize a given uniform affine acyclic oriented matroid of rank 4 have the same number of knots and the same number of links.

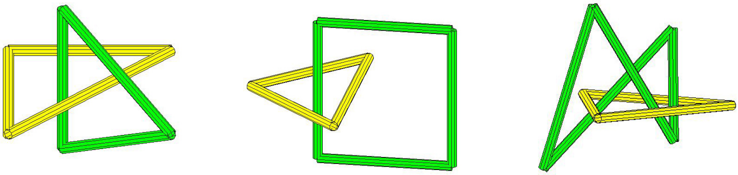
(Proof) A liner embedding of K_7 determines its uniform affine acyclic oriented matroids. Note that uniform affine acyclic oriented matroids of rank 4 on 7 vertices are realizable.

Algorithm to Count Knots and Links in Linear K_7

1. List 462 uniform affine acyclic oriented matroids of rank 4 on 7 vertices,
2. Realize each oriented matroid by choosing 7 points in $\mathbb{R}^3$,
3. For links, we have two methods;
   (a) Check all $(3,3)$-cycles and $(3,4)$-cycles in each realized linear K_7.
   (b) (Without Realizaion) Check four signed circuits conditions for non-trivial links in each matroid.
4. For knots, we have two methods;
   (a) Check all 6-cycles and 7-cycles in each realized linear K_7.
   (b) (Without Realizaion) Check three signed circuits conditions for figure-8 knots in each matroid.

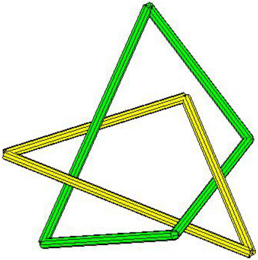
Counting Links in Linear K_7 from Oriented Matroids (1)

- $\mathcal{M}$ contains a (3,3)-Hopf link $\Leftrightarrow$ $(a, b, c, -x, -y) \in \mathcal{M}$, $(a, b, c, -y, -z) \notin \mathcal{M}$ and $(a, b, c, -z, -x) \notin \mathcal{M}$
- $\mathcal{M}$ contains a (3,4)-Hopf link $\Leftrightarrow$ (a) or (b)
  - (a) $(a, b, c, -x, -y) \in \mathcal{M}$, $(a, b, c, -y, -z) \notin \mathcal{M}$, $(a, b, c, -z, -w) \notin \mathcal{M}$ and $(a, b, c, -w, -x) \notin \mathcal{M}$
  - (b) $(a, b, c, -x, -y) \in \mathcal{M}$, $(a, b, c, -y, -z) \in \mathcal{M}$, $(a, b, c, -z, -w) \in \mathcal{M}$ and $(a, b, c, -w, -x) \notin \mathcal{M}$



Counting Links in Linear K_7 from Oriented Matroids (2)

- $\mathcal{M}$ contains a $\{2,4\}$-Torus link $\Leftrightarrow$ (a) and (b)
  - (a) $(a,b,c,-x,-y) \in \mathcal{M}$, $(a,b,c,-y,-z) \notin \mathcal{M}$, $(a,b,c,-z,-w) \in \mathcal{M}$ and $(a,b,c,-w,-x) \notin \mathcal{M}$
  - (b) $(x,y,z,-a,-b) \in \mathcal{M}$ and $(y,z,w,-a,-b) \in \mathcal{M}$ or $(x,y,z,-b,-c) \in \mathcal{M}$ and $(y,z,w,-b,-c) \in \mathcal{M}$ or $(x,y,z,-c,-a) \in \mathcal{M}$ and $(y,z,w,-c,-a) \in \mathcal{M}$



Result for Links in Linear K_7

Number of Links in Linear K_7				K_7 Matroids
(3,3)Hopf	(3,4)Hopf	$\{2,4\}$-Torus	Sum	
7	14	0	21	305
9	18	0	27	97
11	22	0	33	20
13	22	1	36	11
	26	0	39	18
15	26	1	42	7
	30	0	45	3
17	30	1	48	1
Total				462

REMARK. The number of 3-4 Hopf links is not twice the number of 3-3 Hopf links if and only if the number of $\{2,4\}$-torus link is one.

Result for Knots in Linear K_7

Number of Knots in Linear K_7				K_7 Matroids
6-Trefoil	7-Trefoil	Fig-8	Sum	
0	1	0	1	305
1	3	0	4	97
2	5	0	7	20
3	7	0	10	18
	8	1	12	11
4	9	0	13	7
	11	2	17	1
	12	3	19	2
5	11	0	16	1
Total				462

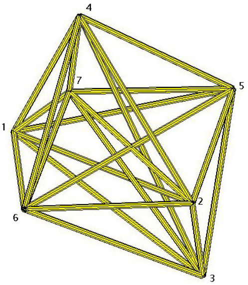
REMARK. No linear K_7 can contain more than five 6-trefoils.

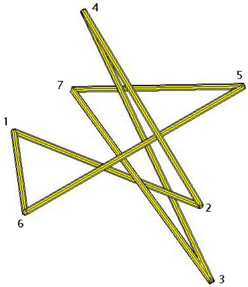
Result for Knots and Links in Linear K_7

<table>
<tr><th colspan="6">Number of Knots and Links in Linear K_7</th></tr>
<tr><th>6-Trefoil</th><th>7-Trefoil</th><th>Fig-8</th><th>(3,3)Hopf</th><th>(3,4)Hopf</th><th>$\{2,4\}$-Torus</th></tr>
<tr><td>0</td><td>1</td><td>0</td><td>7</td><td>14</td><td>0</td></tr>
<tr><td>1</td><td>3</td><td>0</td><td>9</td><td>18</td><td>0</td></tr>
<tr><td>2</td><td>5</td><td>0</td><td>11</td><td>22</td><td>0</td></tr>
<tr><td rowspan="3">3</td><td rowspan="2">7</td><td rowspan="2">0</td><td rowspan="3">13</td><td>22</td><td>1</td></tr>
<tr><td rowspan="2">26</td><td rowspan="2">0</td></tr>
<tr><td>8</td><td>1</td></tr>
<tr><td rowspan="3">4</td><td>9</td><td>0</td><td rowspan="3">15</td><td>26</td><td>1</td></tr>
<tr><td>11</td><td>2</td><td rowspan="2">30</td><td rowspan="2">0</td></tr>
<tr><td>12</td><td>3</td></tr>
<tr><td>5</td><td>11</td><td>0</td><td>17</td><td>30</td><td>1</td></tr>
</table>

REMARK. There are at least ten isotopy classes of embeddings of K_7. If a linear K_7 contains a $\{2,4\}$-torus link then it contains no figure-8 knots.

Examples (1)

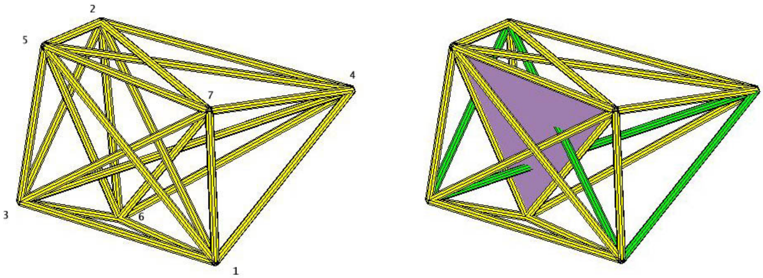




Number of Knots and Links in linear K_7(Matroid-237)					
6-Trefoil	7-Trefoil	Fig-8	(3,3)Hopf	(3,4)Hopf	$\{2,4\}$-Torus
4	12	3	15	30	0

REMARK. In a linear K_7, there exist at most twelve 7-trefoils and at most three figure-8 knots. In tins example, (1, 2, 7, 5, 6, 3, 4), (1, 2, 7, 3, 4, 6, 5) and (1, 2, 4, 3, 7, 5, 6) are figure-8 knots.

Examples (2)



Number of Knots and Links in linear K_7(Matroid-450)					
6-Trefoil	7-Trefoil	Fig-8	(3,3)Hopf	(3,4)Hopf	$\{2,4\}$-Torus
5	11	0	17	30	1

REMARK. In a linear K_7, there exist at most five 6-trefoils and at most one $\{2,4\}$-torus link.

Summary

1. We investigated the number of knots and links in all linear embeddings of K_7.
2. We could see all previous results about the number of knots and links in linear K_7.
3. Further Study
    1. By topological arguments, one can prove some new results obtained by this work.
    2. We are going to investigate knots and links in linear embeddings of K_8

- (Special Thanks) We are grateful to Prof. Youngsik Huh for his valuable comments.

"Thank You!"