Intrinsic linking and knotting in straight-edge embeddings of complete graphs

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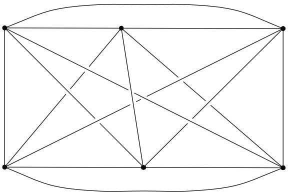
Outline

1. Background
2. Project One: K_6 Links
3. Project Two: K_7 Links
4. Project Three: K_7 Knots
5. Project Four: K_9
6. Further Work

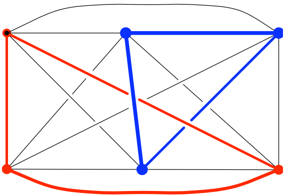
Project One - Started it all...

1983-4: Conway and Gordon, and Sachs:

K_6 is *intrinsically linked*



K_6



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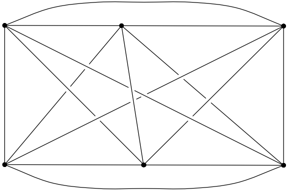

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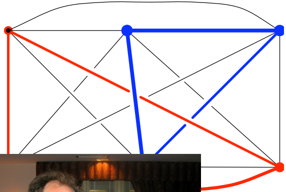
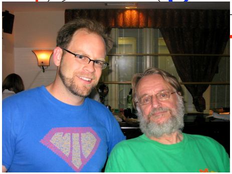


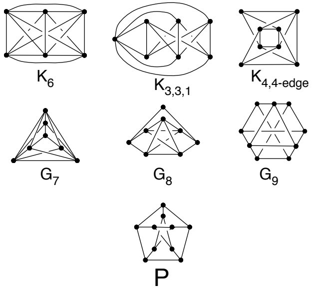


Interesting side note...

Characterization ("Kura-cterization")

1993: Robertson, Seymour, and Thomas:

A graph is *intrinsically linked* iff it contains one of the *Petersen graphs* as a *minor*



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4. *Straight-edge* embeddings of graphs

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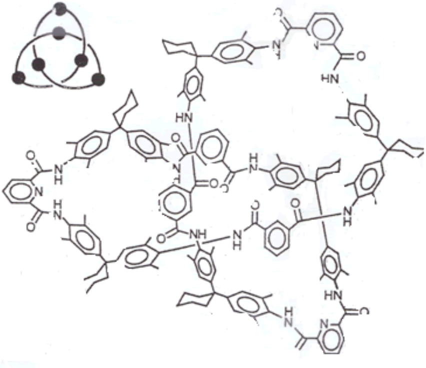

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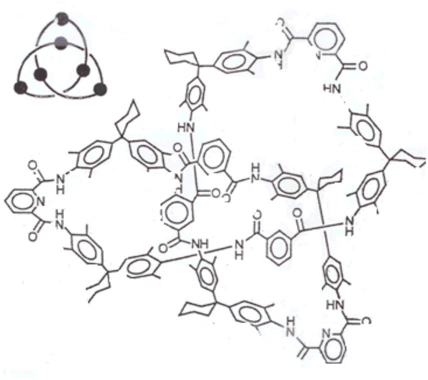

Polyethylene - linear/cyclic, 63 to 78 backbone atoms

Project 1: The motivating question

2004: Workshop with Colin Adams

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(D. Hunt, ONU)
How many linked components occur in a straight–edge embedding of K_6?

Recall, this number must be odd...

Project 1 results

(2006, Hughes and L.)
(2007, Huh and Jeon)

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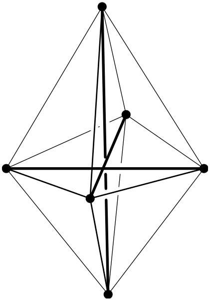
Every *straight-edge* embedding of K_6 has 1 or 3 two-component links

K_6^2: [4,4,4,4,4,4]



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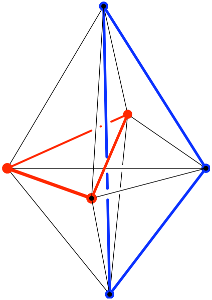

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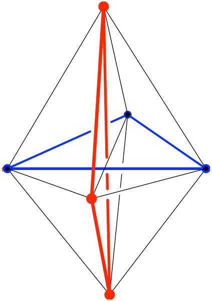

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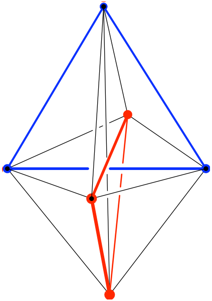

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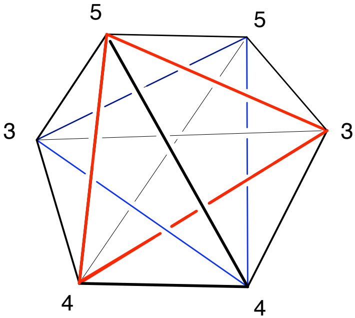
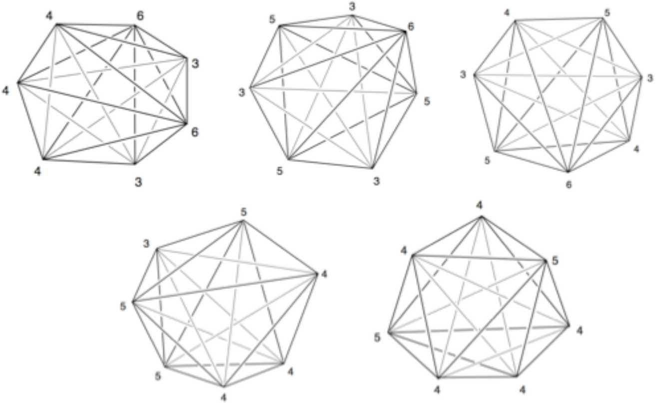
K_6^1: [3,3,4,4,5,5]

Every *straight-edge* embedding of K_6 has 1 or 3 two-component links



Now what?

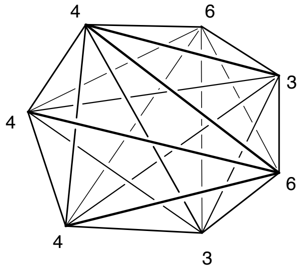
Project 2: 2006: Arbisi and L. (2010)

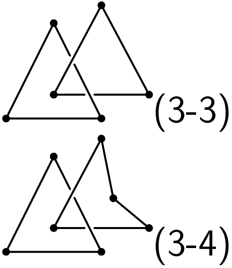


K_7

The good . . .

$$K_7^1$$



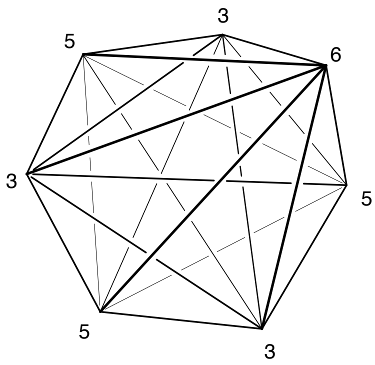


(3-3) links: 7

(3-4) links: 14

The bad ...

$$K_7^2 \quad (K_7^3)$$



(3-3) links: 7 or 9

(3-4) links: 14 or 18

The ugly . . .

The ugly . . . the INTERESTING!

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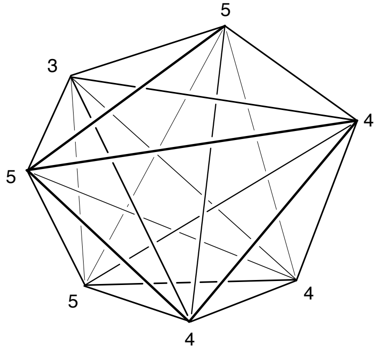
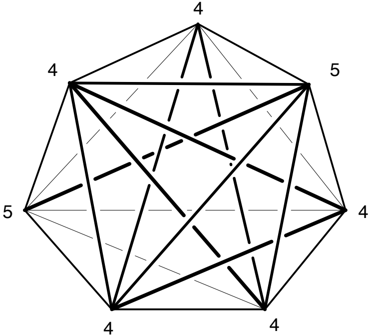
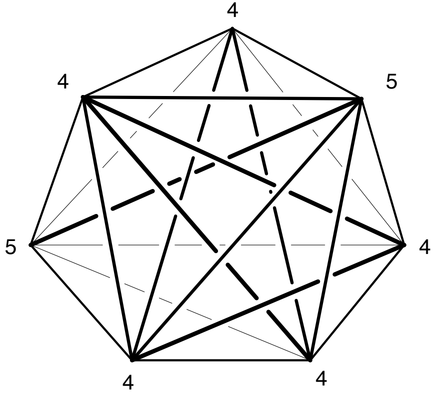
K_7^4



K_7^5



Counting links in K_7^5



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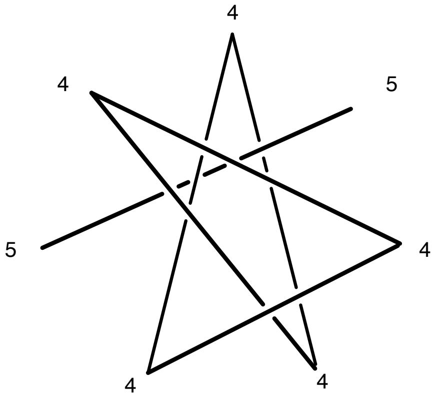

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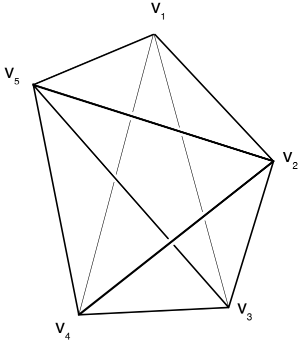

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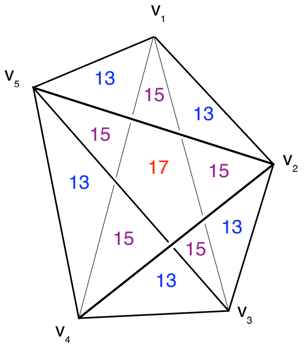
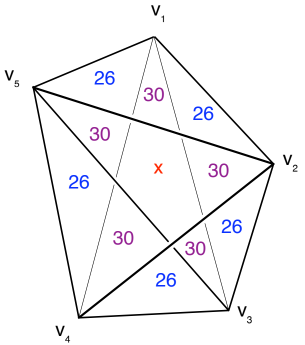


(3-3) links: 13, 15, 17

Counting links in K_7^5



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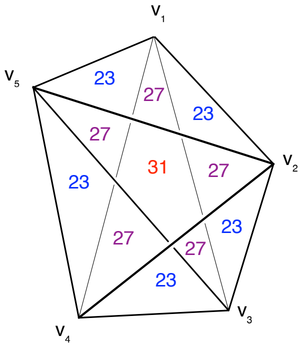
Counting links in K_7^5



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(3-4) links: 23, 27, 31

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Examine larger structures...?

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K_8 has 14 distinct convex hull embeddings, each with a possible

- $\binom{8}{3}\binom{5}{3}=560$ (3-3) links (140)
- $\binom{8}{4}\binom{4}{3}=280$ (3-4) links (70)
- $\binom{8}{4}=70$ (4-4) links
- $\binom{8}{5}=56$ (5-3) links

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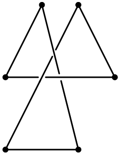
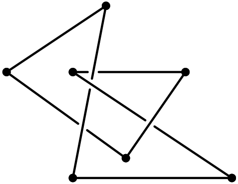
K_9 has 219 distinct convex hulls!

What about knots?

In 1983, Conway and Gordon also showed that K_7 is *intrinsically knotted*.

For K_7, how many possible knots are there?





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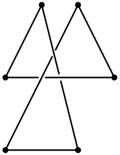

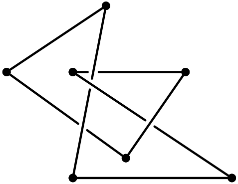

- There are $6!/2 = 360$ Hamiltonian cycles of length 7.
- There are $7 \cdot 5!/2 = 420$ Hamiltonian cycles of length 6.

Project 3 - 2007: Grotheer and L. (2009, Foisy and L.)

Internal Edges	Cycles	Knots	Cycles	Knots	Cycles	Knots	Cycles	Knots	Cycles	Knots
0	14	0	18	0	17	0	24	0	30	0
1	80	0	72	0	92	0	96	0	90	0
2	164	0	174	0	143	0	123	0	120	0
3	88	1	78	1, 3	91	0, 1	90	2, 3	90	1, 2, 3, 4, 5
4	14	0	18	0, 2	16	0, 1, 2	24	0, 1	20	2, 4
5	0	0	0	0	1	0, 1	3	0	10	1, 5
6	0	0	0	0	0	0	0	0	0	0
	K_7^1		K_7^2		K_7^3		K_7^4		K_7^5	

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	K_7^1		K_7^2		K_7^3		K_7^4		K_7^5	

Project Four - 2008: Behrend and L.

Recall we only looked at embeddings where all vertices were on the external hull: two for K_6, five for K_7, fourteen for K_8, and so on...

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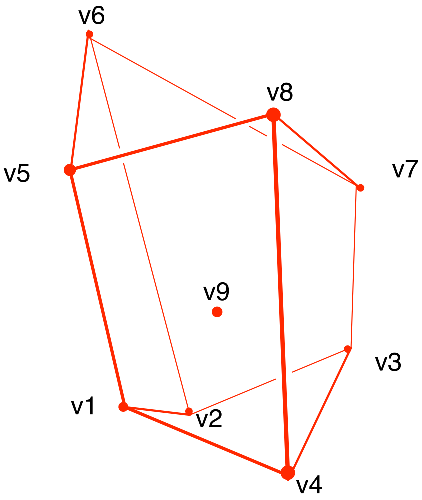
Given K_n with m external vertices and $k = n - m$ internal vertices, is that embedding always ambient isotopic to an embedding with n external vertices?

The idea



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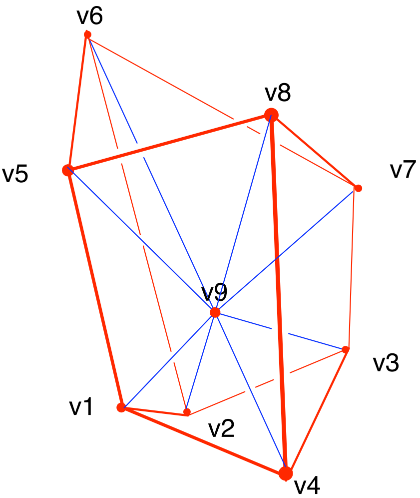

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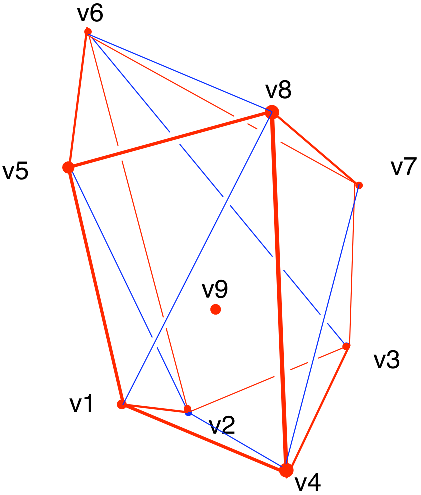

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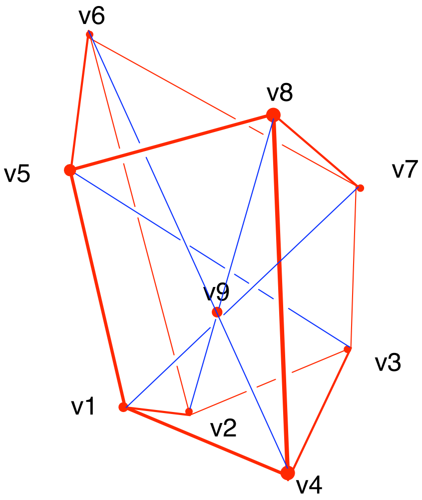

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(2001: Flapan, Naimi, and Pommershein)

K_{10} is intrinsically triple-linked.

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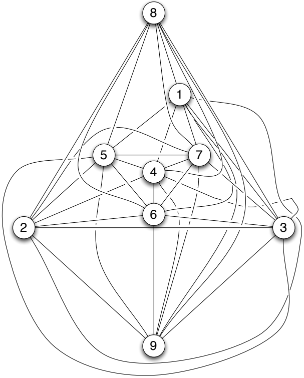

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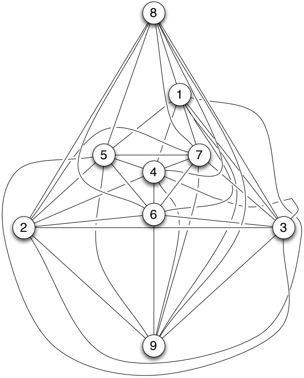

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Thanks. . .

- Colleen Hughes ('06)
- Pam Arbisi ('07)
- Rachel Grotheer ('08)
- Sam Berhend ('09)
- Clay Crocker and Matt Gibson ('13)
- Anderson Research Endowment

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