Bridge position and the representativity of spatial graphs

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Topological graph theory

$$G \subset F$$

Topological spatial graph theory

$$\Gamma \subset F \subset S^3$$

§1 Putting (F, Γ) in an essential Morse position

§2 Constructing spatial graphs with arbitrarily high representativity

§3 High representativity implies high complexity

§4 Strong spatial embedding conjecture

§1 Putting F in an essential Morse position

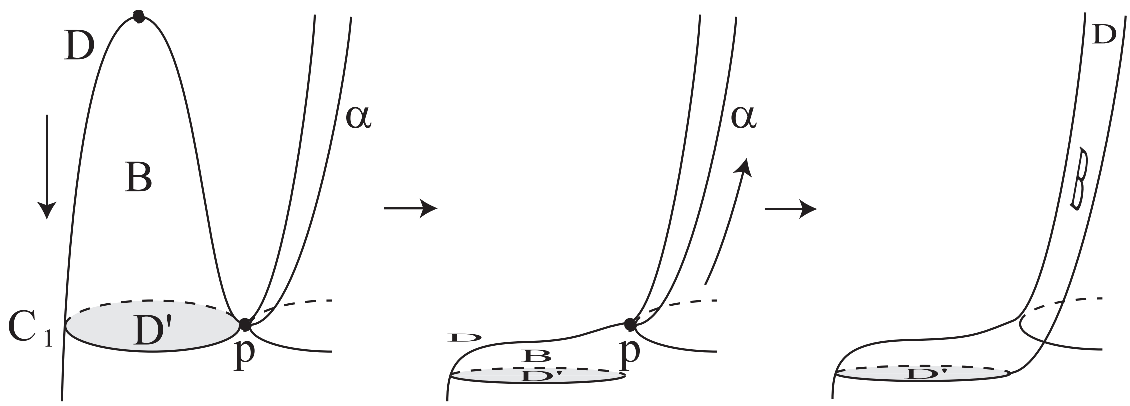
Let Γ be in a bridge position with respect to the height function $h : S^3 \to \mathbb{R}$.

Let F be a closed surface containing Γ.

Then there exists an isotopy of (F, Γ) in S^3 such that it keeps the bridge position of Γ and F has no inessential saddle point (i.e. essential Morse position).

Lemma 1

(F, Γ) can be put in an essential Morse position.



In case that F is a 2-sphere

By Lemma 1, F has no saddle point, and hence only one maximal/minimal point.

Thus F intersects a bridge sphere S in a single loop.

Theorem 1

Let Γ be put in a bridge position.
Then Γ is trivial if and only if there exists a 2-sphere F containing Γ such that F intersects the bridge sphere S in a single loop.

It is worth to notice that an isotopy of unknotting Γ can be decomposed into two isotopies by any bridge sphere S.

Theorem 1 extends Otal's result that any non-minimal bridge position of the trivial knot is stabilized.

In case that F has a positive genus

By Lemma 1, F has only essential saddle points.

Then there exists a bridge level sphere S such that $S \cap F$ contains at least two essential loops.

Since there are at least two essential innermost loops l_1 and l_2 of $S \cap F$ in S, which bound compressing disks D_1 and D_2 for F in S,

$$|\partial D_1 \cap \Gamma| + |\partial D_2 \cap \Gamma| \leq |\Gamma \cap S|$$

Definition 1

We define the *representativity* of (F, Γ) as

$$r(F, \Gamma) = \min_{D \in \mathcal{D}_F} |\partial D \cap \Gamma|$$

where $\mathcal{D}_F$ is the set of all compressing disks for F in S^3.

Definition 2

We define the *representativity* of Γ as

$$r(\Gamma) = \max_{F \in \mathcal{F}} r(F, \Gamma)$$

where $\mathcal{F}$ is the set of all closed surfaces of positive genus containing Γ.

Definition 3

We define the *bridge string number* of Γ as

$$bs(\Gamma) = \min_{\Gamma \in \mathcal{BP}_\Gamma} |\Gamma \cap S|$$

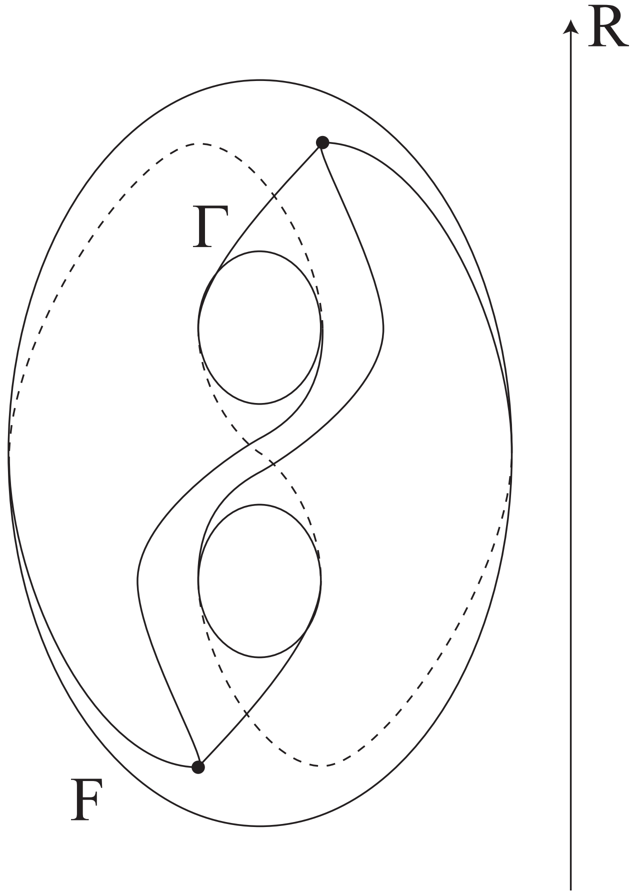
where $\mathcal{BP}_\Gamma$ is the set of all bridge position of Γ.

Remark. $r(\Gamma) \geq 1$ and $bs(\Gamma) \geq \beta_1(G) + 1$ for any spatial graph Γ

Theorem 2

$$r(\Gamma) \leq \frac{bs(\Gamma)}{2}$$

Example 1.



$$2 = r(F, \Gamma) \leq r(\Gamma) \leq \frac{bs(\Gamma)}{2} \leq \frac{5}{2}$$

Hence $r(\Gamma) = 2$.

Example 2.

- $2 \leq r(K) \leq b(K)$ for any non-trivial knot K

- $r(K) = \min\{p, q\}$ for a (p, q)-torus knot K

- $r(K) = 2$ for a 2-bridge knot K

Theorem 3

- $r(K) \leq 3$ for an algebraic knot K
- $r(K) = 3$ for a (p, q, r)-pretzel knot K if and only if $(p, q, r) = \pm(-2, 3, 3)$ or $\pm(-2, 3, 5)$

Conjecture 1

$r(K) = 2$ for an alternating knot K

§2 Constructing spatial graphs with arbitrarily high representativity

Lemma 2

For a Heegaard surface of positive genus F in S^3 and for any integer $n \geq 2$, there exists a knot K non-separatingly contained in F such that $r(F, K) = n$.

Lemma 3 (Fox's reimbedding theorem)

A connected compact 3-dimensional submanifold of S^3 can be reimbedded in S^3 so that it is the complement of a union of handlebodies in S^3.

Lemma 4 (Bonahon)

An irreducible compact 3-manifold with boundary has a unique characteristic compression body.

Theorem 4

For any closed surface F with $g(F) \geq g(G)$ and for any integer n, there exists a spatial graph Γ of G contained in F such that $r(F, \Gamma) \geq n$.

Put $S^3 = M_1 \cup_F M_2$.

First by Lemma 4, we take a characteristic compression body V_i for F in M_i.

Next by Lemma 3, we reimbed $V_1 \cup_F V_2$ in S^3 so that it is the complement of a union of handlebodies in S^3.

Then F becomes a Heegaard surface in S^3 and by Lemma 2, it contains a knot K such that $r(F, K) = n$.

Finally we embed G in the Heegaard surface F so that it contains K, and restore the Fox's reimbedding.

Satellite construction

$$S^3 = M_1 \cup_F M_2 \supset V_1 \cup_F V_2 \supset F \supset \Gamma \supset K$$

$$\downarrow \text{Fox}$$

$$S^3 = M_1' \cup_{F'} M_2' \supset V_1 \cup_{F'} V_2 \supset F' \supset \Gamma' \supset K'$$

Key ingredients:

- F' is a Heegaard surface of S^3.
- $r(F', \exists K') = n$ in S^3 by Lemma 2
- $F' \supset \exists \Gamma' \supset K'$
- $r(F, K) = r(F', K') \geq n$ in $V_1 \cup_F V_2 = V_1 \cup_{F'} V_2$
- $r(F, K) \geq n$ in S^3
- $r(F, \Gamma) \geq r(F, K) \geq n$

§3 High representativity implies high complexity

Theorem 5

If $r(\Gamma) > \beta_1(G)$, then Γ contains a connected totally knotted spatial subgraph.

Γ is *totally knotted* if $\partial N(\Gamma)$ is incompressible in $S^3 - \Gamma$.

Remark. Totally knotted spatial graphs are non-free.

Theorem 6

If $r(\Gamma) = n$, then Γ is spatially n-connected.

Γ is *spatially n-connected* if it has no essential tangle decomposing sphere S with $|\Gamma \cap S| < n$.

Proof of Theorem 5.

Lemma 5

If Γ is primitive, then $bs(\Gamma) \leq 2\beta_1(G)$.

Proof: Let T be a spanning tree of Γ.
Then $\partial N(T)$ is a bridge sphere for Γ such that $|\partial N(T) \cap \Gamma| = 2\beta_1(G)$.
Hence $bs(\Gamma) \leq 2\beta_1(G)$.

Lemma 6

If Γ is not primitive, then Γ contains a connected totally knotted spatial subgraph.

Proof: Let T be a spanning tree of Γ such that Γ/T is a non-trivial bouquet.
Then Γ/T contains a minimally knotted (hence totally knotted) subgraph Γ_0/T_0.
Hence Γ contains a connected totally knotted spatial subgraph Γ_0.

Proof of Theorem 6.

Let F be a closed surface containing Γ such that $r(F, \Gamma) = n$.

Suppose that there exists an essential tangle decomposing sphere S for Γ with $|\Gamma \cap S| < n$.

We may assume that F intersects S in loops, and assume that $|F \cap S|$ is minimal.

Then the innermost disk D in S bounded by an innermost loop of $F \cap S$ is a compressing disk for F since S is an essential tangle decomposing sphere.

It follows that $r(F, \Gamma) \leq |\partial D \cap \Gamma| \leq |\Gamma \cap S| < n$.

§4 Strong spatial embedding conjecture

Definition 4

The (maximal) *representativity* of a non-planar graph G is defined as

$$r(G) = \max_{F \in \mathcal{F}} \min_{C \in \mathcal{C}_F} |C \cap G|,$$

where $\mathcal{F}$ is the set of all closed surfaces containing G and $\mathcal{C}_F$ is the set of all essential loops in F.

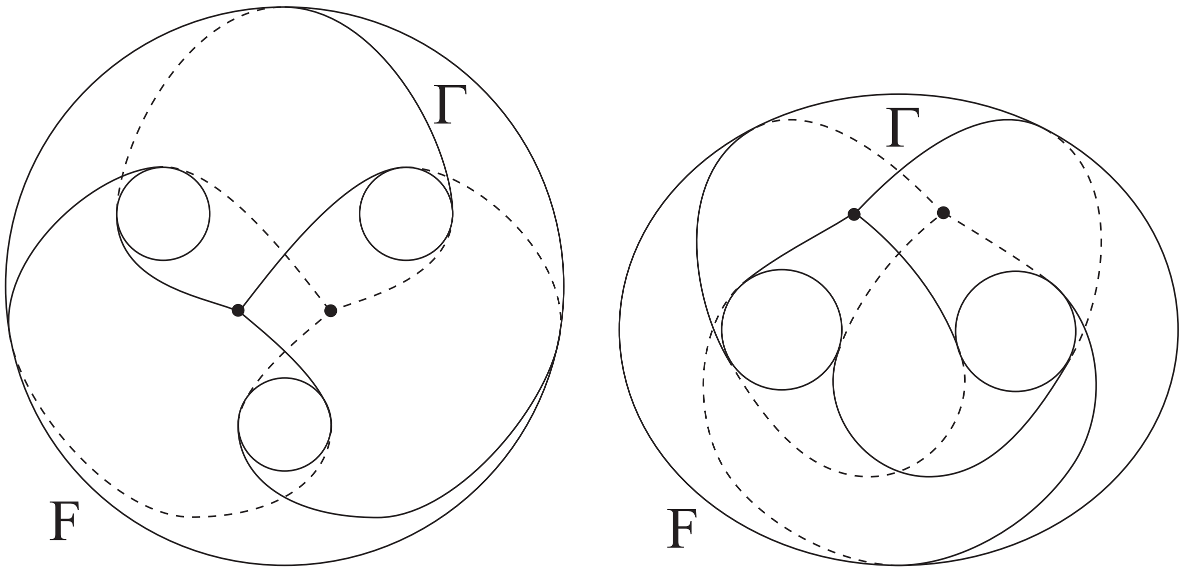
Strong embedding conjecture

For a 2-connected non-planar graph G, $r(G) \geq 2$.

Strong spatial embedding conjecture

For a non-trivial spatial graph Γ of a 2-connected graph G, $r(\Gamma) \geq 2$.

Example 3.



Kinoshita's theta curve on two closed surfaces

We have $r(\Gamma) = 2$ like Example 1.

Theorem 7

If θ_n-curve Γ_i $(i = 1, 2)$ satisfies the strong spatial embedding conjecture, then some connected sum $\Gamma_1 \# \Gamma_2$ also satisfies the strong spatial embedding conjecture.