A knotted handlebody and a spatial graph

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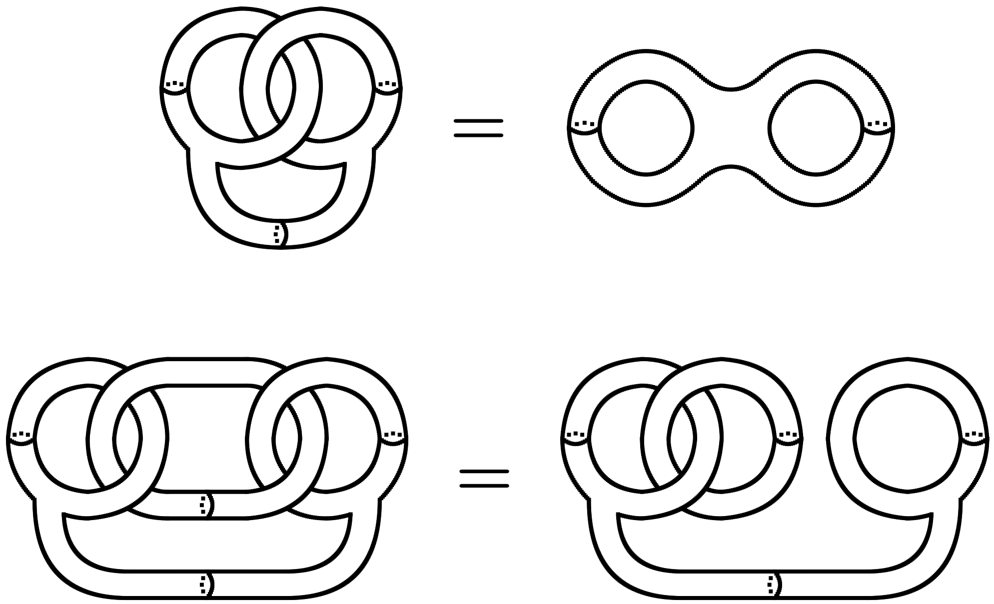
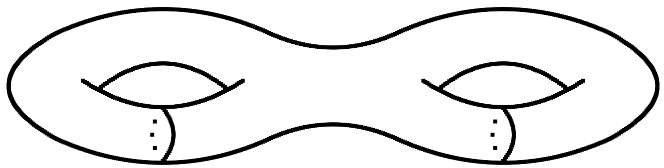


What is a knotted handlebody?

A knotted handlebody, or a handlebody-knot,

is a handlebody embedded in a 3-dimensional manifold M.

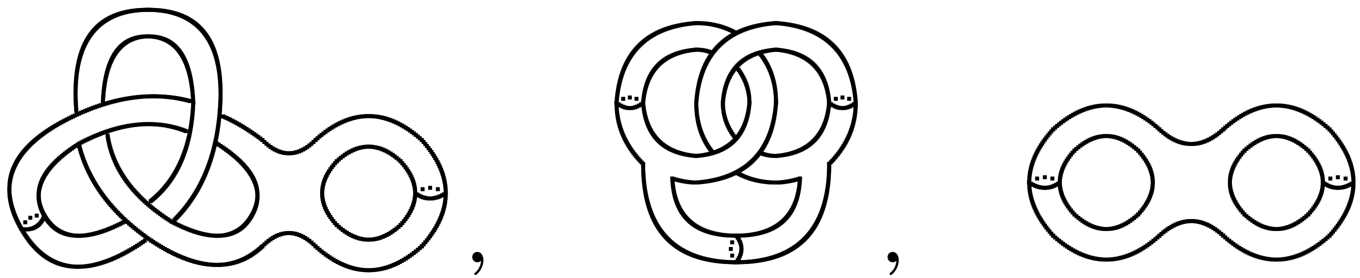
(In this talk, $M := S^3$.)



$\hookrightarrow S^3$

handlebody

Example

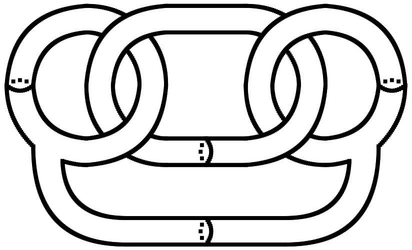
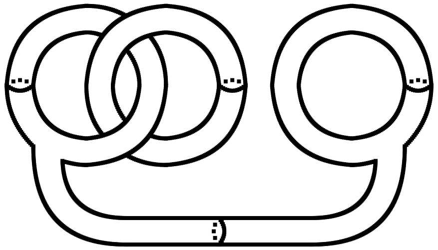


$, \quad , \quad , \ldots$

A handlebody-link

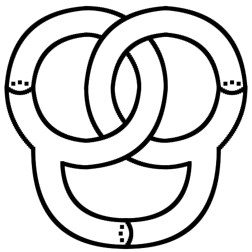
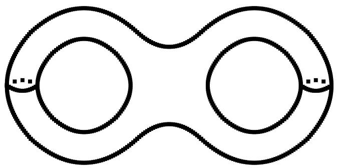
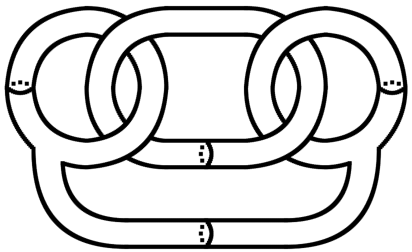
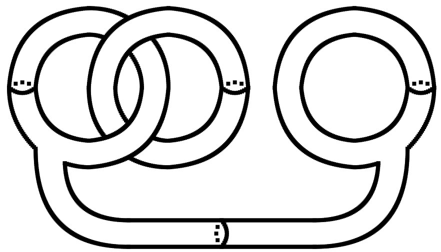
is a disjoint union of handlebodies embedded in S^3.

Example

 ,  , . . .

Def H, H': handlebody-links

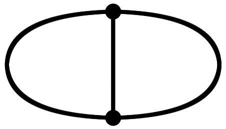
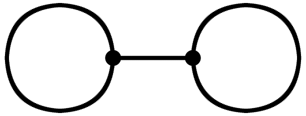
$$H = H' \overset{\text{def}}{\Leftrightarrow} \exists \text{ isotopy } f_t : S^3 \to S^3 \text{ s.t. } f_0 = \mathrm{id}_{S^3},\ f_1(H) = H'$$

 =     = 

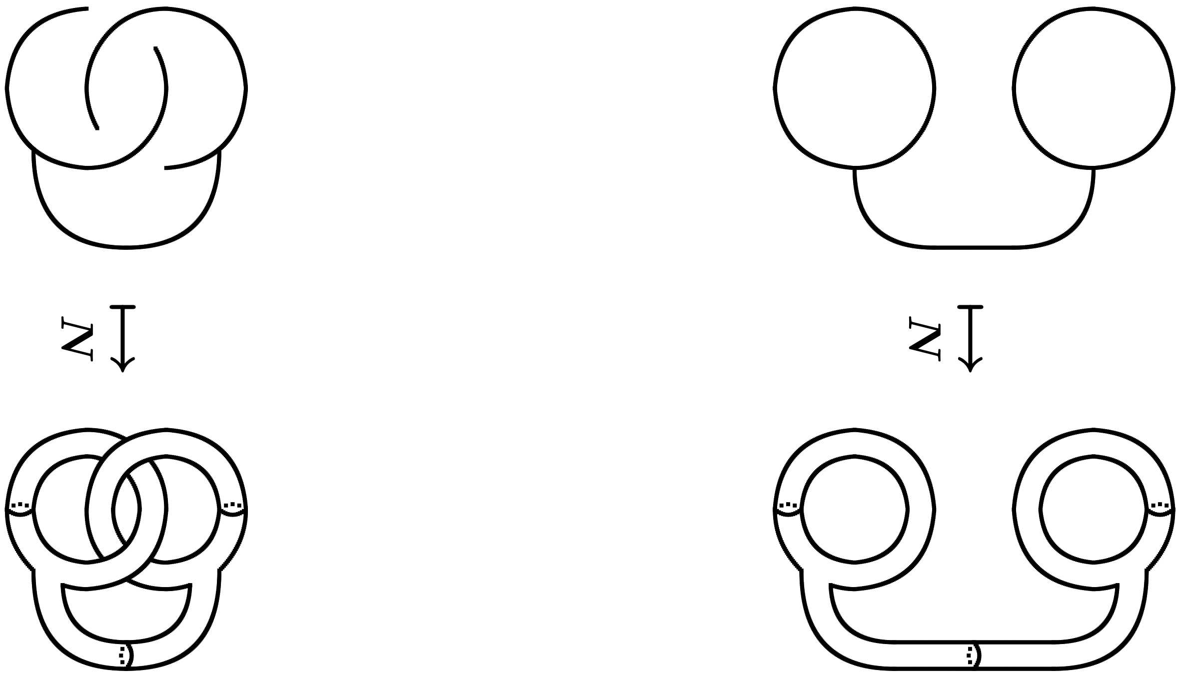
A spatial graph and a handlebody-link

<u>A spatial graph</u>

is a finite graph embedded in S^3.

( $\hookrightarrow S^3$,  $\hookrightarrow S^3, \dots$)

<u>Example</u>



<u>Lem</u> $N : \{\text{spatial graph}\} \longrightarrow \{\text{handlebody-link}\}$ is surjective.

$K \longmapsto N(K)$ (a reg. nbd. of K)

Lem $N : \{\text{spatial graph}\} \longrightarrow \{\text{handlebody-link}\}$ is surjective.

$$K \mapsto N(K) \text{ (a reg. nbd. of } K\text{)}$$

A spatial spine

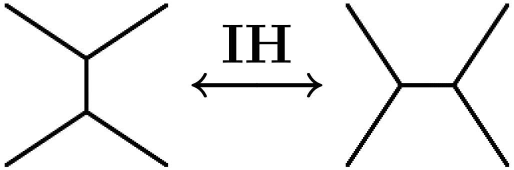
is a trivalent graph embedded in S^3.

Lem $N : \{\text{spatial spine}\} \longrightarrow \{\text{handlebody-link}\}$ is surjective.

$$K \mapsto N(K)$$

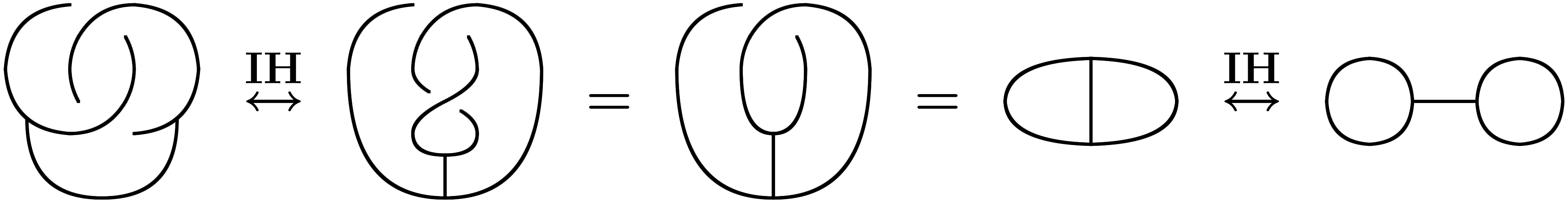
- What is the kernel of this map?

An IH-move

is the local move 

on spatial spines.

Example



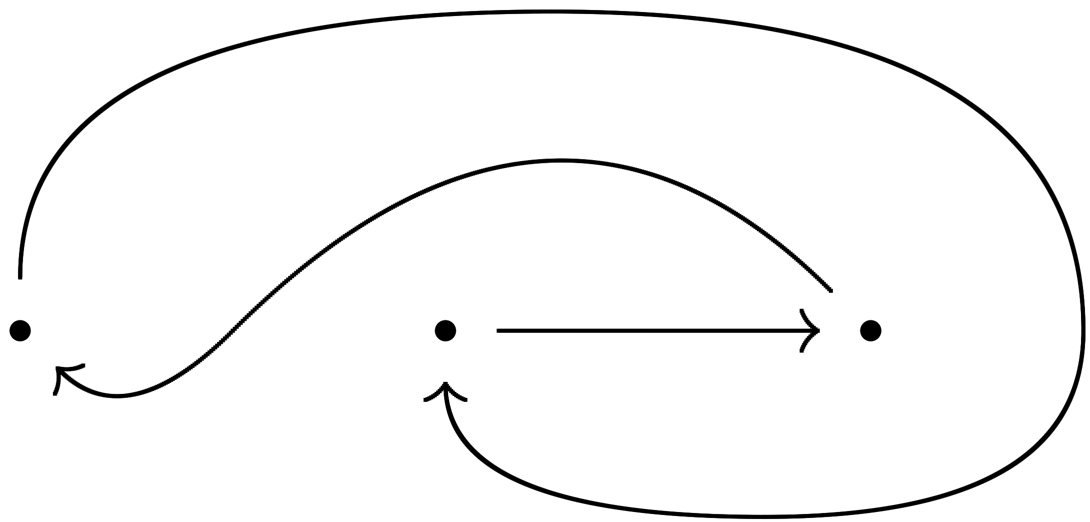
Thm [I] K, K': spatial spines

$$N(K) = N(K') \Leftrightarrow K \overset{\mathrm{IH}}{\leftrightarrow} K_1 \overset{\mathrm{IH}}{\leftrightarrow} K_2 \overset{\mathrm{IH}}{\leftrightarrow} \cdots \overset{\mathrm{IH}}{\leftrightarrow} K',$$

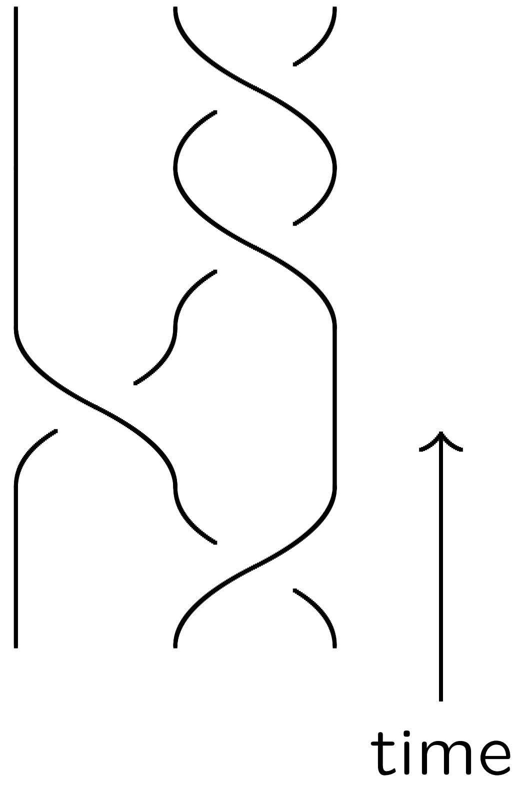
that is, $\{$handlebody-link$\} = \{$spatial spine$\}/$IH-move.

Motions of particles and IH-moves

a motion of particles

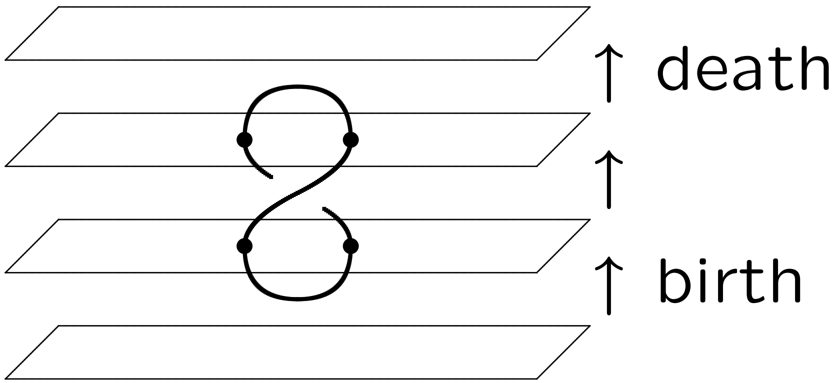


a braid

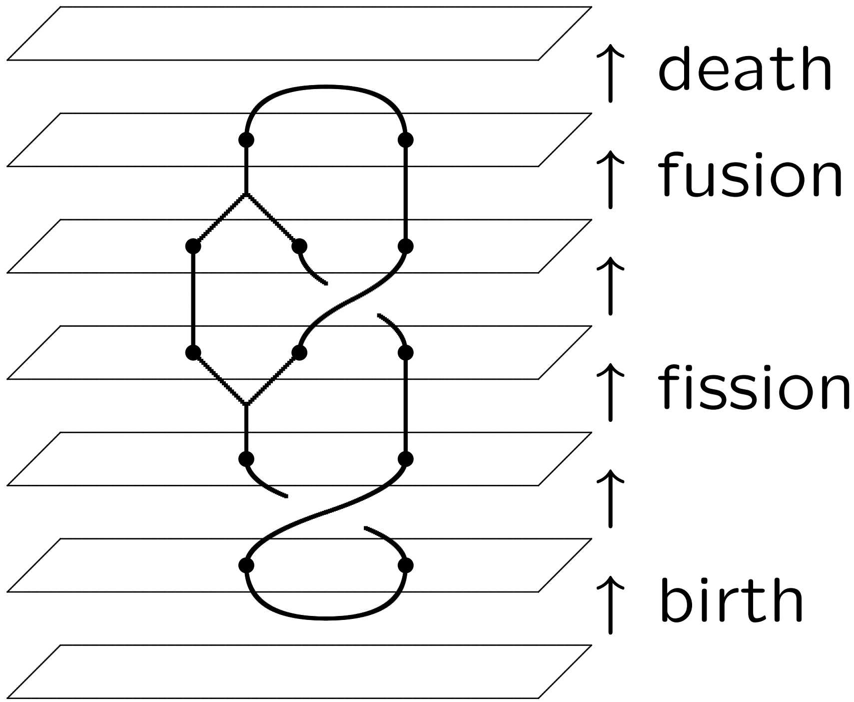


• a braid = a motion of particles

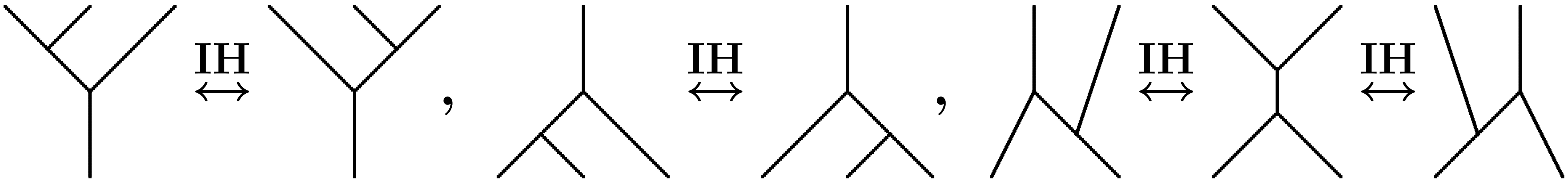
a link



a spatial graph



Various IH-moves



- a handlebody-link = a motion not depending on the local order of fission and fusion

The neighborhood equivalence relation

<u>Def</u> [Suzuki]

Spatial graphs K, K' are neighborhood equivalent ($K \sim_N K'$)
$\overset{\text{def}}{\Leftrightarrow}$ $\exists$ ori. pres. homeo. $h : S^3 \to S^3$ s.t. $h(N(K)) = N(K')$

<u>Remark</u>

- $K \sim_N K' \Leftrightarrow N(K) = N(K')$ as handlebody-links
- $K = K' \Rightarrow K \sim_N K' \Rightarrow E(K) \approx E(K')$ (exterior)

(1) $K \neq K'$ and $K \sim_N K'$

(2) $K \not\sim_N K'$ and $E(K) \approx E(K')$

(3) $E(K) \not\approx E(K')$

(1) $\Rightarrow$ IH-complex [I-Kishimoto]

(2) $\Rightarrow$ invariants for handlebody-links [I-Iwakiri, I-Jang-Oshiro]

(3) $\Rightarrow$ invariants for exteriors (the fundamental group, etc.)

Invariants for handlebody-links

- the fundamental group, etc.
- quandle cocycle invariants
  - with commutative flows [I-Iwakiri]
  - with noncommutative flows [I-Jang-Oshiro]

Step to defining a quandle cocycle invariant

(1) a flow (a $\mathbb{Z}_2$-flow in this talk)

(2) a quandle coloring invariant (a p-coloring in this talk)

(3) a quandle cocycle invariant

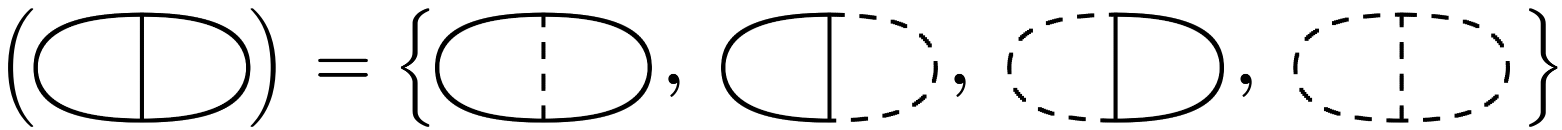
A $\mathbb{Z}_2$-flow of a spatial ($\mathbf{3}$-regular) graph $\boldsymbol{K}$

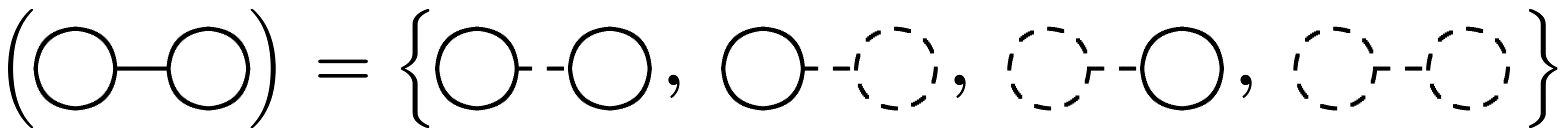
is an assignment of $\mathbf{0}$ or $\mathbf{1}$ to each edge of $\boldsymbol{K}$ such that $\mathbf{1}$-edges form a constituent link of $\boldsymbol{K}$.

$\mathbf{0}$-edge $\quad$ $\mathbf{1}$-edge

$\mathbf{Flow}(\boldsymbol{K}) := \{\varphi \,|\, \mathbb{Z}_2\text{-flow of } \boldsymbol{K}\}$.

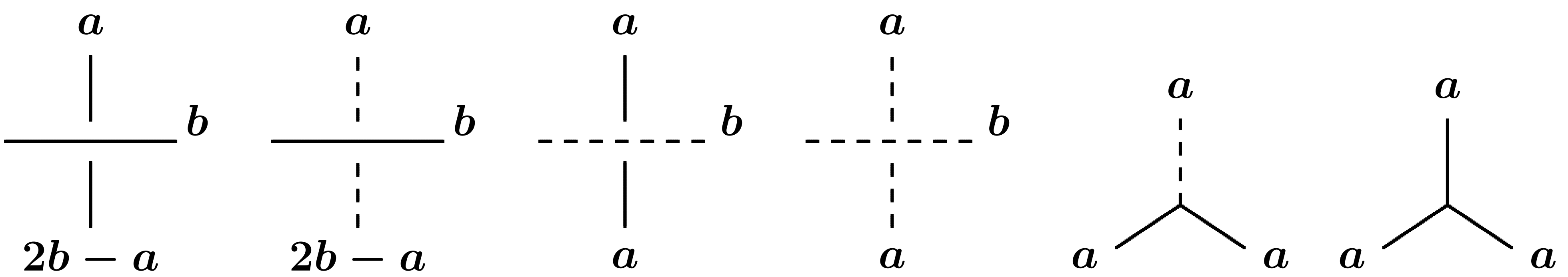
Example

Flow


Flow


<u>A p-coloring of a diagram D_φ of a flowed spatial graph (K, φ)</u>

is an assignment of an element of $\mathbb{Z}_p = \{0, \ldots, p-1\}$ to each arc of D_φ satisfying the following.

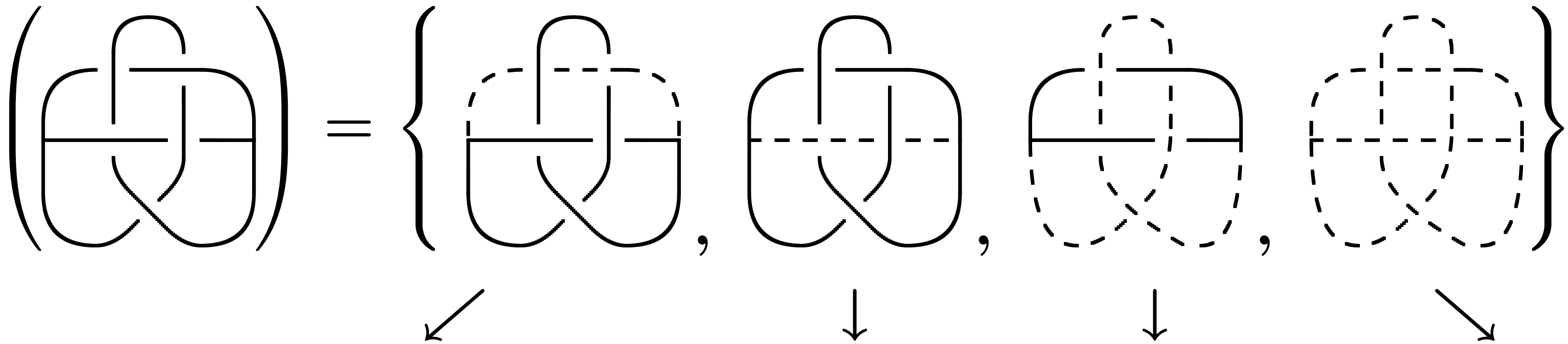


$Col_p(D_\varphi)$: the set of p-colorings of D_φ

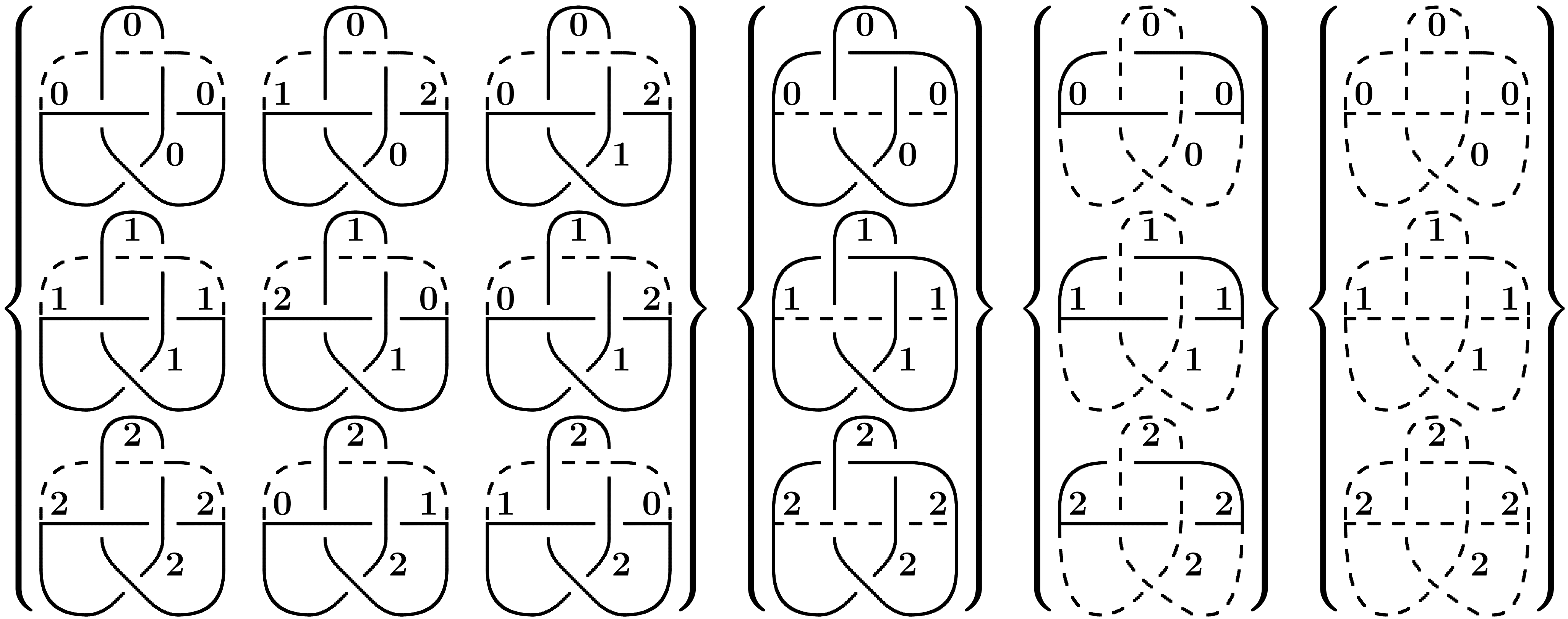
<u>Thm</u> The multiset $\{\#Col_p(D_\varphi) \mid \varphi \in \mathrm{Flow}(K)\}$ is an invariant of the handlebody-link $N(K)$ (and a spatial graph K).

Example ($p = 3$)

Flow



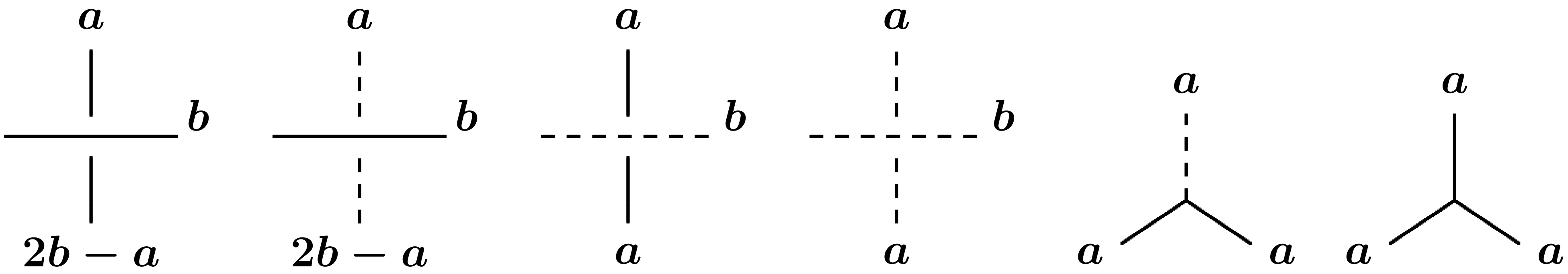
$\underline{Col_3}$

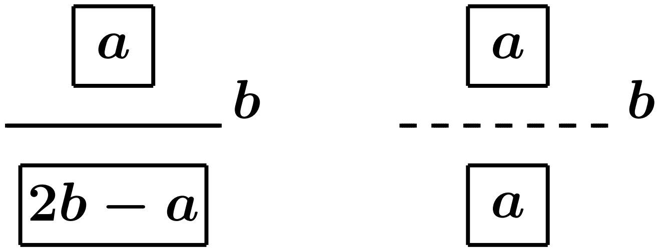


The coloring invariant is $\{9, 3, 3, 3\}$. cf. $\{3, 3, 3, 3\}$ (trivial)

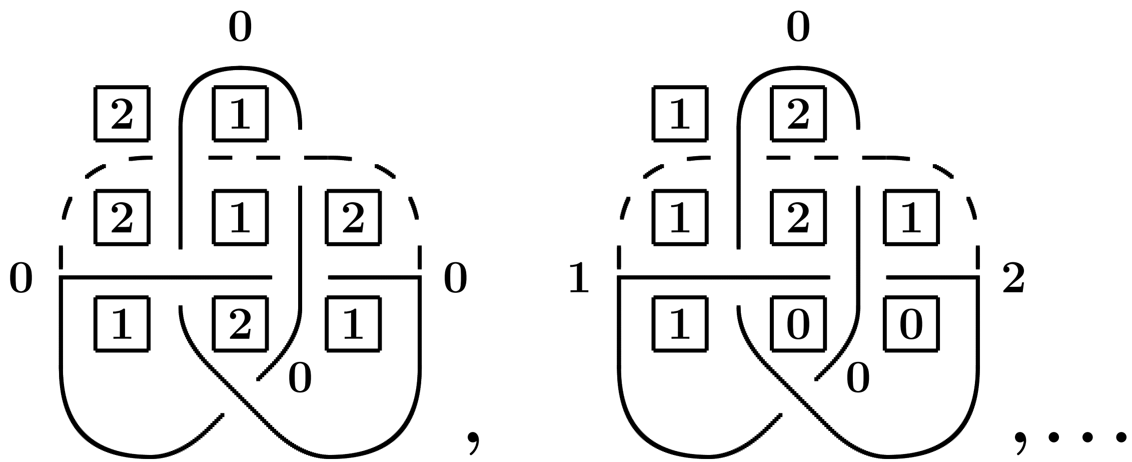
A shadow p-coloring of a diagram D_φ

is an assignment of an element of $\mathbb{Z}_p = \{0, \ldots, p-1\}$ to each arc and region of D_φ satisfying the following.





Example ($p = 3$)

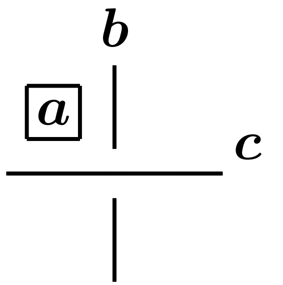


$\widetilde{Col}_p(D_\varphi)$: the set of shadow p-colorings of D_φ

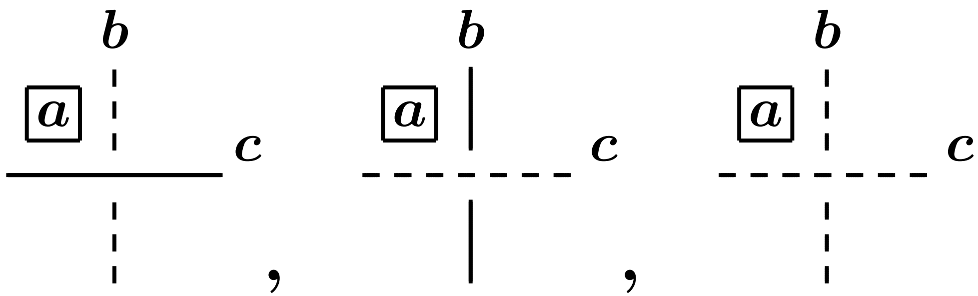
The quandle cocycle invariant Φ_p

$$\Phi_p(K) := \sum_{\varphi \in \mathrm{Flow}(K)} \prod_{C \in \widetilde{Col}_p(D_\varphi)} x_{B(C)},$$

where $B(C)$ is the sum of the Boltzmann weights on all crossings.

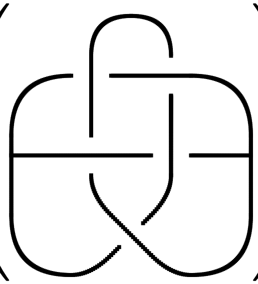
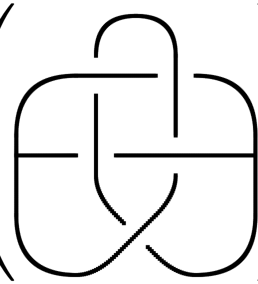


$\xrightarrow{\text{Boltzmann weight}}$ $(a-b)\dfrac{(2c-b)^p + b^p - 2c^p}{p} \in \mathbb{Z}_p$



$\xrightarrow{\text{Boltzmann weight}}$ 0

Example

$\Phi_3\Bigg($  $\Bigg) = x_0^9 x_2^{18} + 3x_0^9 \neq x_0^9 x_1^{18} + 3x_0^9 = \Phi_3\Bigg($  $\Bigg)$