Similarities between flat and planar graphs

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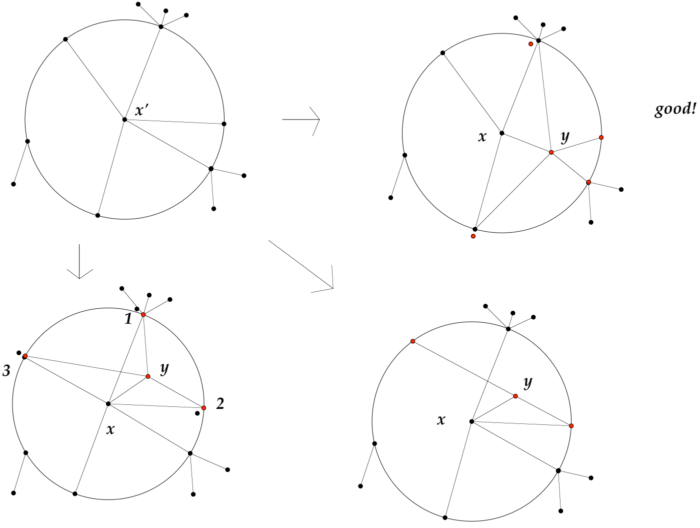
- Consider a graph G, with $n+1$ vertices, such that G does not contain K_5 nor $K_{3,3}$ as a minor.
- Key Lemma (Thomassen): There exists an edge $e=(x,y)$ of G such that G/e is 3-connected. As G/e has n vertices, by the induction hypothesis, it can be embedded in the plane. Moreover, as $G-x'$ is $2-$connected, the graph of G/e near x' looks like a cycle C with spokes emanating from x'.

- Then expand the vertex x' back into (x, y), the result is a planar embedding of G', unless K_5 or $K_{3,3}$ is a minor.



bad--has K_5 minor

bad--has $K_{3,3}$ minor

- Observation: if there is an $S^0 = \{v, w\}$ on the cycle C such that $C - \{v, w\}$ is two components C_1, C_2, and every neighbor of x is contained in $C_1 \cup \{v, w\}$ and every neighbor of y is contained in $C_2 \cup \{v, w\}$, then G will be planar.

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- Once the $3-$connected case has been established, it's easy to establish result for lower connectivity graphs.

Sachs' Linkless Embedding Conjecture

We call a graph G *flat* if there exists a spatial embedding of G, $\phi(G)$, such that for every cycle C in G, there is a disk $D \subset \mathbb{R}^3$ such that $D \cap \phi(G) = C$.

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Recall that the Petersen Family of graphs consists of K_6 and the six other graphs obtained from K_6 by $\Delta - Y$ and $Y - \Delta$ exchanges.

Outline of steps involved in generalizing Thomassen's proof of Kuratowski's Theorem to a (possible??!!!!!) proof of Sachs' linkless embedding conjecture:

Assume by induction that every 3$-$connected graph on n or fewer vertices that does not contain a Petersen Family graph as a minor has a flat embedding.

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Note: it's well known that a minor-minimal non flat graph must be $3-$connected.

Start with a $3-$connected graph, G, that does not contain a Petersen Family graph as a minor, and has $n+1$ vertices. Contract an edge $e = (x, y)$, such that the resulting graph is $3-$connected (such an edge exists by Thomassen's lemma).

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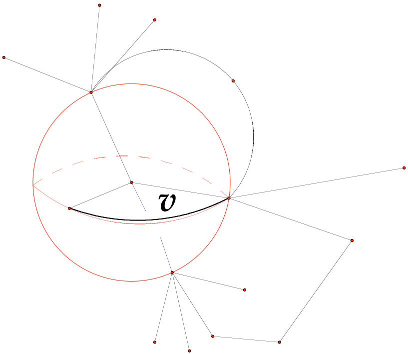
The resulting graph has no minor in the Petersen family, and thus by the induction hypothesis has a flat embedding. Call the vertex that results from the contracted edge v. Call $G/e = H$.

Take a flat embedding of H, $\phi(H)$. Ambient isotope the embedding so that all edges that are incident to v are straight line segments of unit length, and the rest of H lies outside the unit sphere centered at v, except that edges between neighbors of v lie on the unit sphere.



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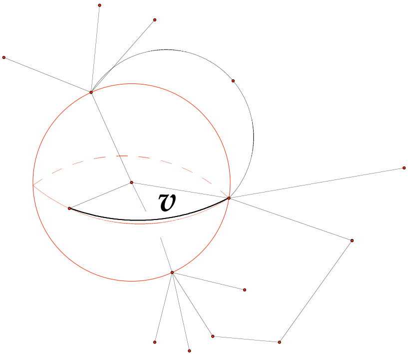

By abuse of notation we call this new (but equivalent) embedding $\phi(H)$.

(Note: we can do this by Bohme's lemma: the cycles containing v in the subgraph induced by v and its neighbors form a collection of cycles whose pair-wise intersection is always connected.) Note that the subgraph induced by the neighbors of v must be planar, as $\phi(H)$ is flat.

Recall that a *disk assignment* D_ϕ for a flat spatial graph $\phi(G)$ is a collection of disks, $\{D_i | i \in I\}$ such that there are I cycles of G, and for each cycle C_i in G, D_i panels C_i. That is, $D_i \cap \phi(G) = C_i$.

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Given $\phi(H)$, we consider two disk assignments $D_\phi = \{D_i\}$ and $E_\phi = \{E_i\}$ to be *equivalent* if, for each $i \in I$, there is an ambient isotopy of $\phi(H) \cup D_i$ with $\phi(H) \cup E_i$ such that $\phi(H)$ is fixed.

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The resulting intersections form edges. We call these edges *slice edges*. If a slice edge is parallel to an existing edge in H, we discard the slice edge in the slice graph.

Note that slice edges will not intersect edges of H, though they may intersect each other. When they do intersect, we get new vertices (*slice vertices*) in the slice graph. Given a disk assignment, we pick representatives so that there are as few intersections (slice vertices) as possible.

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- adjacent slice vertices.

Conjecture: Under these conditions, $S(\phi(H), D_\phi, v)$ is a uniquely determined graph.

To reiterate, $S(\phi(H), D_\phi, v)$ is the graph consisting of the induced subgraph formed by v and its neighbors, as well as the slice edges and vertices.

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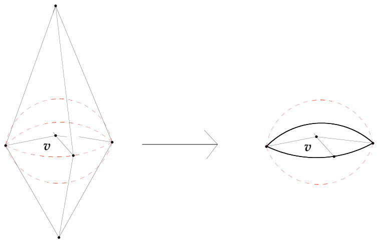
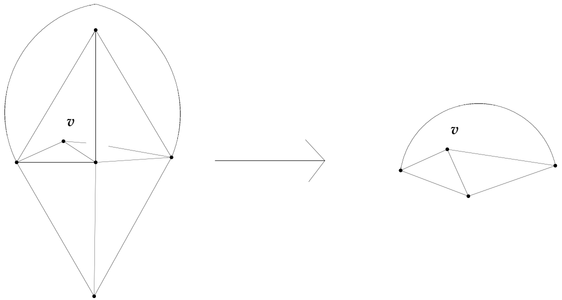
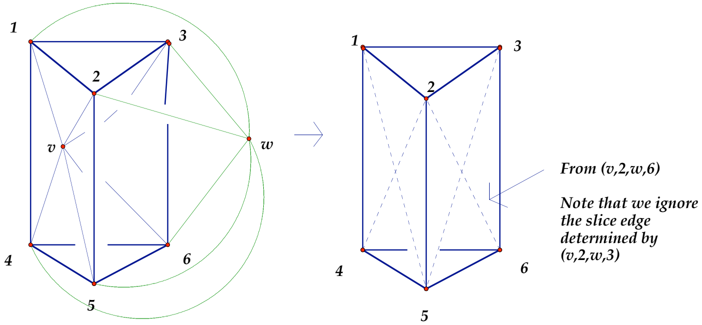
S comes with an embedding inherited from $\phi(H)$. We call the embedding $\phi(S)$.



An example of a slice graph.

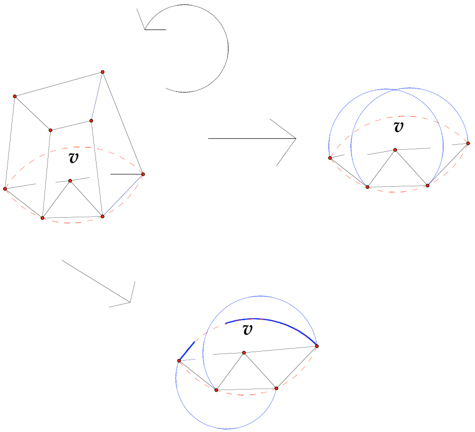


A second example of a slice graph.

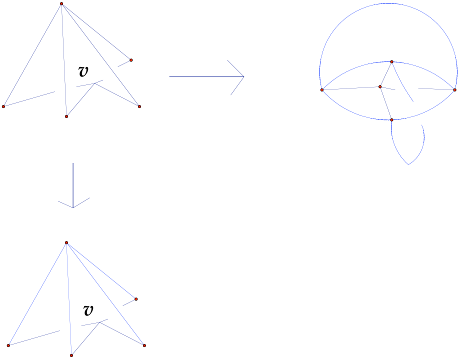


The maximally flat graph from Bohme and an associated neighbor graph.

- An example of $\phi(H)$ with different disk assignments leading to different S.



Another example of $\phi(H)$ with different disk assignments leading to different S.



If we restrict to a single disk assignment (up to equivalence at v), and minimize slice vertices, then we reiterate our conjecture: S is unique.

Now, similarly to Thomassen's proof, we try to expand, in $\phi(H)$ (and in S), the vertex v back to the edge (x, y), and see if we get a flat embedding. We call the graph resulting from S by expanding v into (x, y) an *expanded slice graph*, denoted S'. We denote the embedding that results from expanding the vertex in the (?) natural way, $\phi(S')$.

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(We hope!) The slice graph at a vertex is analogous to the wheel with spokes in Thomassen's proof. In some sense, the slice graph only considers local information. It cuts out information from the graph embedding that is irrelevant, as far as flatness goes.

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Proof sketch: (<–) Start with the flat $\phi(S')$. Extend to an embedding of G by adding the edges of $\phi(H)$ that lie outside the unit sphere. What results is an embedding of G with some slice edges (from S') added on the unit sphere. Clearly, any cycle that lies outside the open unit ball can be paneled.

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Conjecture: The resulting embedding of G is flat.

The (- ->) direction: Sketch of possible proof: if every disk assignment (or perhas if just a “minimal bridge disk assignment") leads to a non-flat $\phi(S')$, then G must contain a Petersen graph as a minor....

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Just worry about possible expanded slice graphs. This makes things easier.

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We call the graph S with this information on the vertices the *marked slice graph*, $m(S)$.

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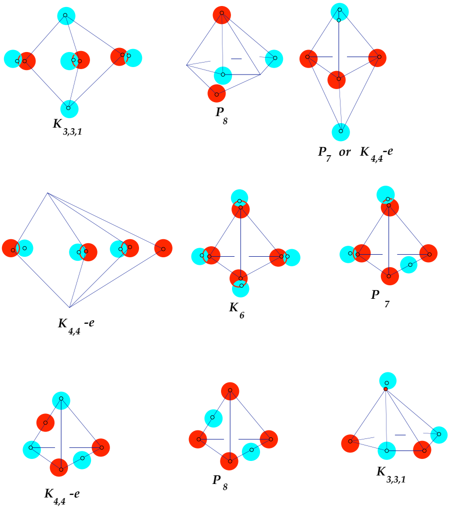
Thanks to RST, there are nine minor-minimal such graphs. They were obtained by taking G/e with G in the Petersen family, where e is an edge or non-edge of G. (All Y's were turned into triangles....) :

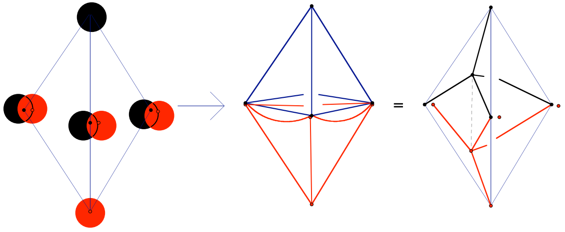


$K_{3,3,1}$
P_8
P_7 or $K_{4,4}$-e
$K_{4,4}$-e
K_6
P_7
$K_{4,4}$-e
P_8
$K_{3,3,1}$

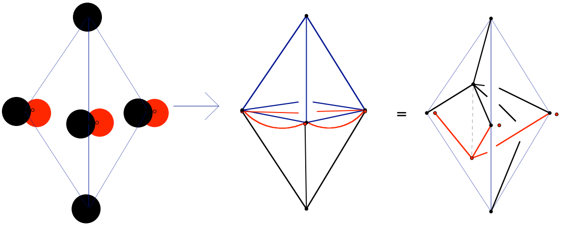
Last big gap: come up with a direct argument that these 9 cases cover all minor-minimal $m(S)$ that do not lead to a flat S'.

Claim/Conjecture:

The graph S' is flat if there exists a disk assignment for $\phi(S)$ for which there is a separating circle for $m(S)$.

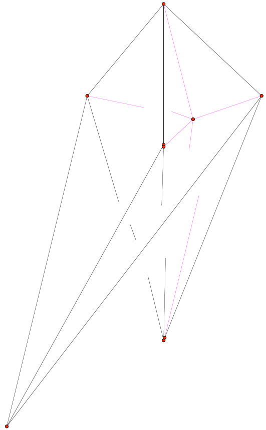


FLAT!

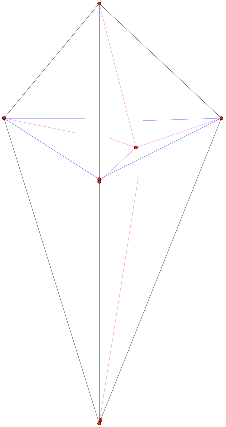


NOT FLAT!

Let's follow this reasoning for an edge contraction on $K_{4,4} - e$. Contract an edge between a degree 3 vertex and a degree 4 vertex. Call the resulting graph H. Here is a flat embedding of H:



The associated S is:



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