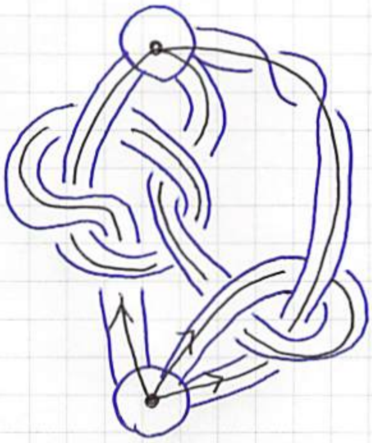
① disk/band surface (Kauffman - Simon - Wolcott - Zhao, '93)



For spatial graph,
disk/band surface is ori. surface constructed as follows:
1. attach disks on vertices
2. connect disks along edge by bands

Remark. ambiguity comes from only twisting on bands

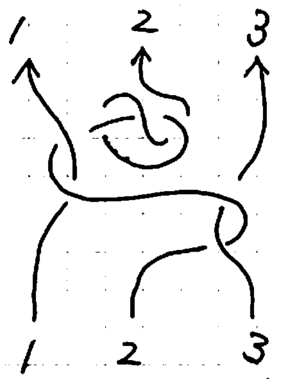
Theorem (K-S-W-Z, '93)
$G = \theta$ or $K_4 \Rightarrow \forall$emb. of G,
$\exists!$ d/b surface S
s.t. Seifert linking form

$$H_1(S) \times H_1(S) \to \mathbb{Z}$$
$$(x, y) \mapsto lk(x^+, y)$$

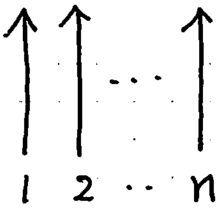
is zero

This implies that we can control twisting by Seifert linking form.

② String link : embedded arcs in the cylinder s.t. i th component connects bottom i and top i



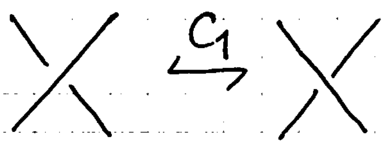
3-string link



: n-string trivial link, denote it by $\mathbb{1}_n$ or $\mathbb{1}$

③ C_k-move (Habiro, '94, '00)

C_k-move ($k \geq 2$) : as



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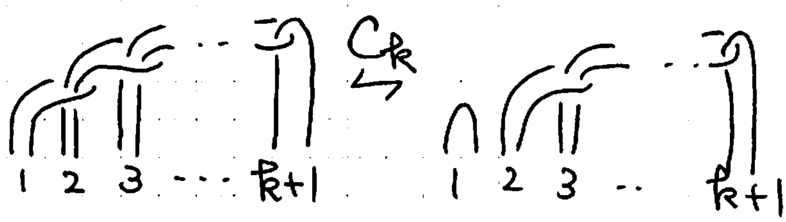


$\overset{C_k}{\sim}$: C_k-equi is equi relation generated by C_k-moves

Remark. C_{k+1}-equi implies C_k-equi.

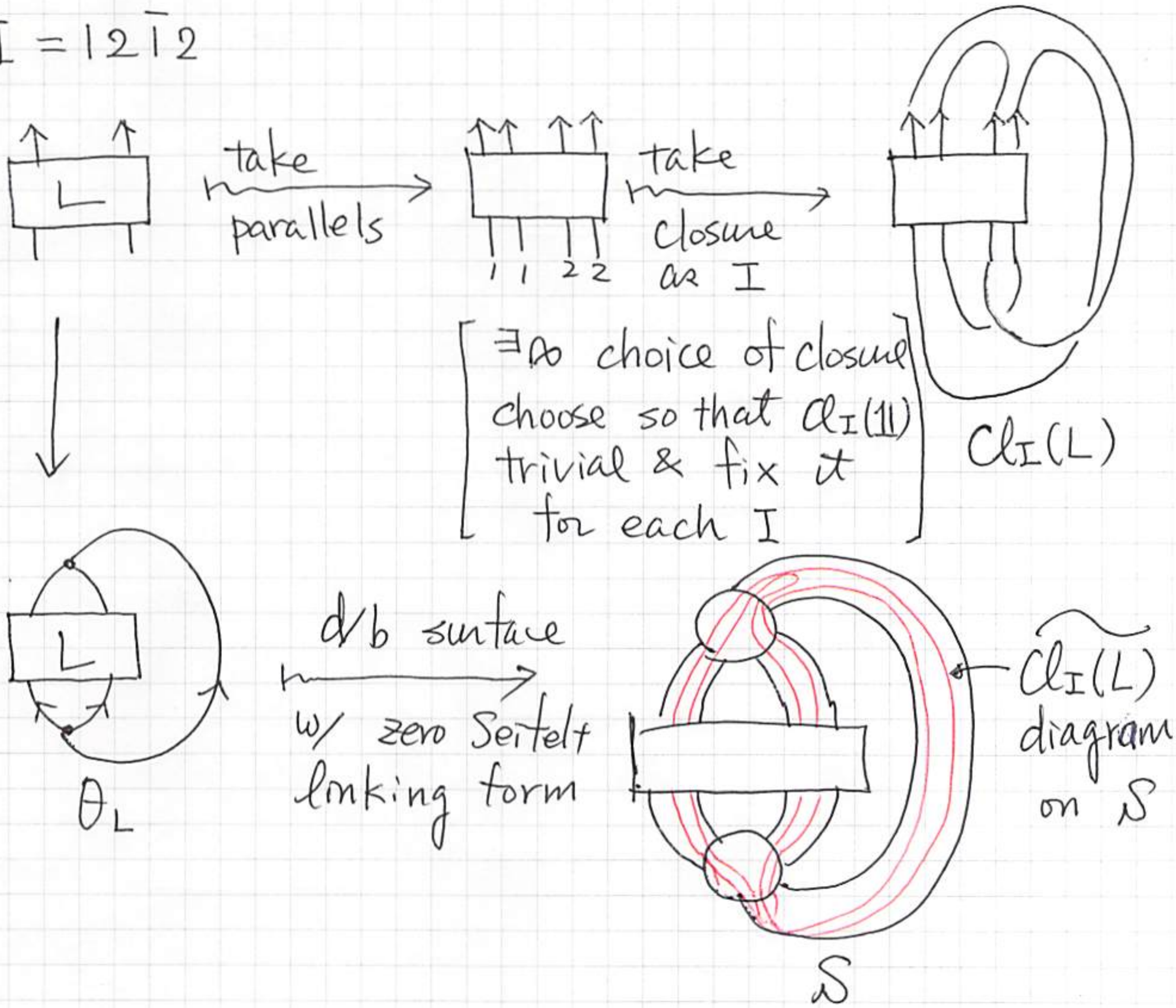
④ closure inv.

L : 2-string link with $\mu_L(12) = 0$

$I = i_1 \cdots i_m$: seq. of $\{1, 2, \bar{1}, \bar{2}\}$

$Cl_I(L)$ is defined as:

$I = 12\bar{1}2$



Note: $Cl_I(L)$ is inv for both L & θ_L

Now we have the following observation

$k \geq 2$, L, L' : 2-string links w/ $\mu(12)=0$

$$\begin{array}{llll} L \overset{C_k}{\sim} L' & \Rightarrow Cl_I(L) \overset{C_k}{\sim} Cl_I(L') & \underset{\text{Gusarov-Habiro}}{\Leftrightarrow} & v(Cl_I(L)) = v(Cl_I(L')) \\ & \text{for } \forall I & & \text{for } \forall I \ \& \\ \Downarrow & \nearrow & & \forall \text{order} \leq k-1 \\ \theta_L \overset{C_k}{\sim} \theta_{L'} & & & \text{Vassiliev inv. } v \end{array}$$

?

If ? holds, $L \overset{C_k}{\sim} L' \Leftrightarrow \theta_L \overset{C_k}{\sim} \theta_{L'}$

<u>Remarks</u> (1) Same observation between spatial K_4 & 3-string links is given

(2) (Nikkuni-Y)
? is true for $k \leq 5$, θ & K_4
Hence C_k-classifications of spatial θ · K_4 are given by closure inv.

(3) By [Soma-Sugai-Y, '97], we have similar observation between string links & emb. of planar, prime, trivalent graphs.