Achirality and linking numbers of links

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[ABSTRACT]

An ordered and oriented n-component link L in the 3-sphere is said to be achiral if it is ambient isotopic to its mirror image ignoring the orientation and ordering of the components. For an ordered and oriented n-component link L, let λ_L be the product of linking numbers of all 2-component sublinks in L. For $n = 4m+3$, where m is a non-negative integer, we show that if L is achiral then $\lambda_L = 0$. And for $n \neq 4m+3$, we show that there exists an n-component achiral link L with $\lambda_L \neq 0$ by using achiral embeddings of complete graphs with n vertices K_n.

Contents

§1. Introduction

Definition

Let $L = K_1 \cup K_2 \cup \cdots \cup K_n$ be an oriented and ordered n-component link in 3-sphere $\mathbb{S}^3$. A link L is said to be achiral if there exists an orientation-reversing self homeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$ such that $f(L) = L$ that ignores orientations and ordering.

- $\lambda_{ij} = \ell k(K_i, K_j)$ $(i < j, i, j = 1, 2, \ldots, n)$
- $\lambda_L = \prod_{i<j, i,j=1,2,\ldots,n} \lambda_{ij}$

Theorem 1

In the case of $n = 4m + 3$ (m is a non-negative integer), if L is achiral as an unordered unoriented n-component link, then $\lambda_L = 0$.

Theorem 2

In the case of $n \neq 4m + 3$ (m is a non-negative integer), there exists an n-component achiral link with $\lambda_L \neq 0$.

§2. Results of Flapan and Weaver for achiral embeddings of complete graphs

K_n : a complete graph

$\tilde{K}_n$: an embedding of K_n

Theorem 3 [Flapan and Weaver, 1992]

In the case of $n \neq 4m + 3$ (m is a non-negative integer), K_n is achirally embeddable.

Sketch proof)

- G : K_{4m}
- P : a plane in $\mathbb{R}^3$
- C : a circle on P
- ℓ : a perpendicular line
- $\alpha : G \to G$: an order 4 automorphism of G such that the orbit of every vertex under α has length 4

Place the vertices of G on C so that the $90°$ rotation about ℓ induces the same permutation of vertices as α.

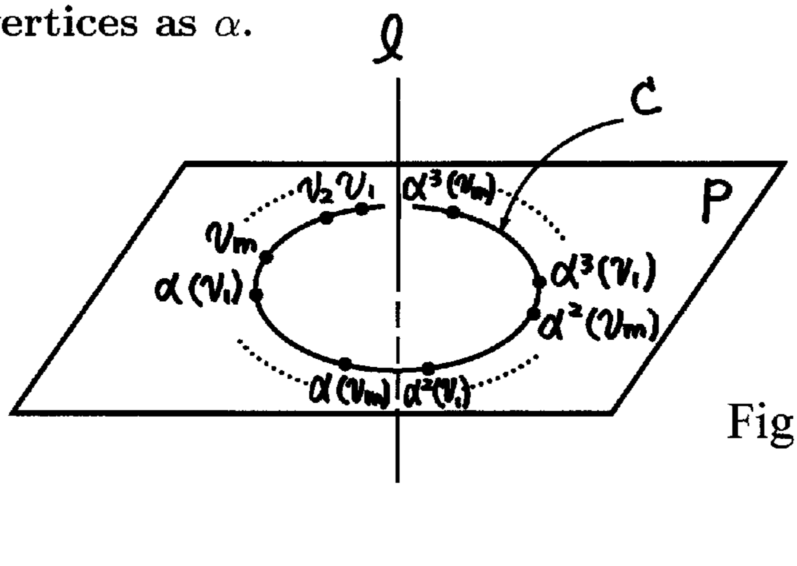

Fig. 1

- $f : \mathbb{R}^3 \to \mathbb{R}^3$: **the composition of the** $90°$ **rotation about** ℓ **with the reflection through the plane** P
- $e_1, \ldots, e_p$: **representatives from each edge orbit**
- $E_1, \ldots, E_p$: **ellipsoids such that symmetric about** ℓ **and** P **and meet at the circle** C **containing the vertices**
- E_i^+ : **the upper half-ellipsoid of** E_i
- E_i^- : **the lower half-ellipsoid of** E_i

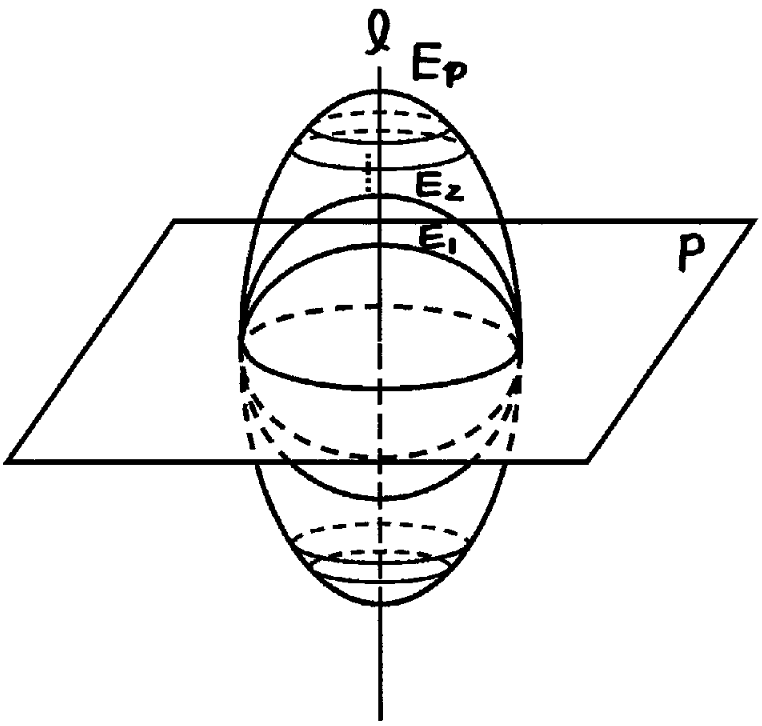

Fig. 2

$(1)\alpha^2(e_i) = e_i$

Embed e_i in E_i^+ so that it is invariant under the $180°$ rotation f^2 about ℓ. And embed $\alpha(e_i)$ as the image of this edge under f, contained in E_i^-

$(2)\alpha^2(e_i) \neq e_i$

Let vertices v and w be the ends of e_i. Consider the semicircles A and B of C with end points v and $\alpha^2(v)$, the antipodal point of v. Without loss of generality, v and w are both contained in A. $\alpha^2(v)$ and $\alpha^2(w)$ are both contained in B. Embed e_i in E_i^+ so that it is disjoint from its image under f^2. Embed $\alpha(e_i), \alpha^2(e_i), \alpha^3(e_i)$ as the images of e_i under f, f^2, f^3 respectively.

Then we have an achiral embedding $\tilde{K}_{4m}$.

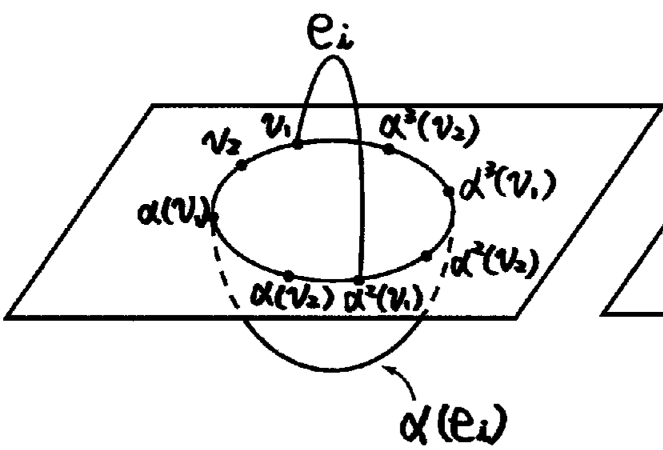

Fig. 3 $\alpha^2(e_i) = e_i$
(In the case of m=2)

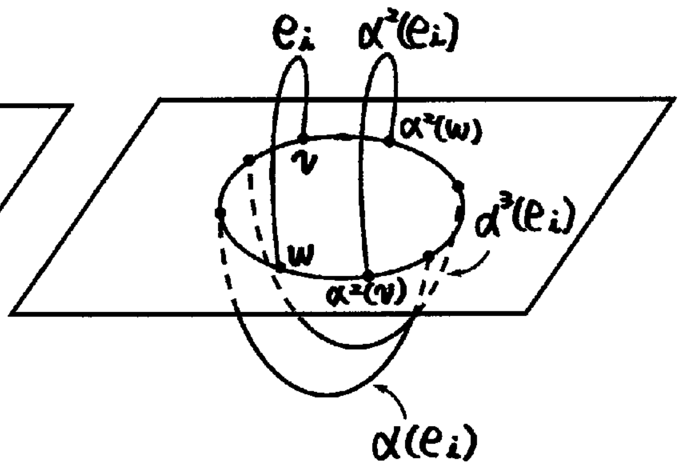

Fig. 4 $\alpha^2(e_i) \neq e_i$
(In the case of m=2)

For K_{4m+1}, first we embed a subgraph K_{4m} of K_{4m+1} achirally as stated above. Then add the final vertex at the point where the plane P intersects a line ℓ, with straight edges connecting this vertex to all the other vertices. This embedding is achiral.

For K_{4m+2}, embed subgraph K_{4m} of K_{4m+2} as stated above. Let E_0 be an additional symmetric ellipsoid that is contained in the interior of all the other E_i. Then add the remaining two vertices at $\ell \cap E_0^+$ and $\ell \cap E_0^-$. Connect these last two vertices to each other by a line segment in ℓ and to the other $4m$ vertices by intersections of $E_0^\pm$ with vertical planes. We have an achiral embedding of K_{4m+2}.

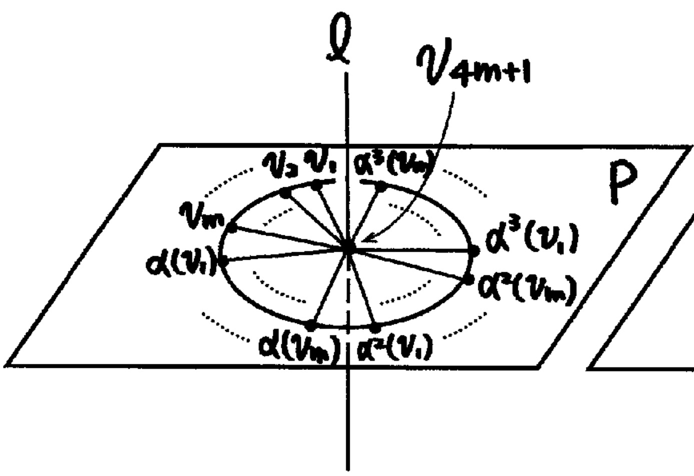

Fig. 5

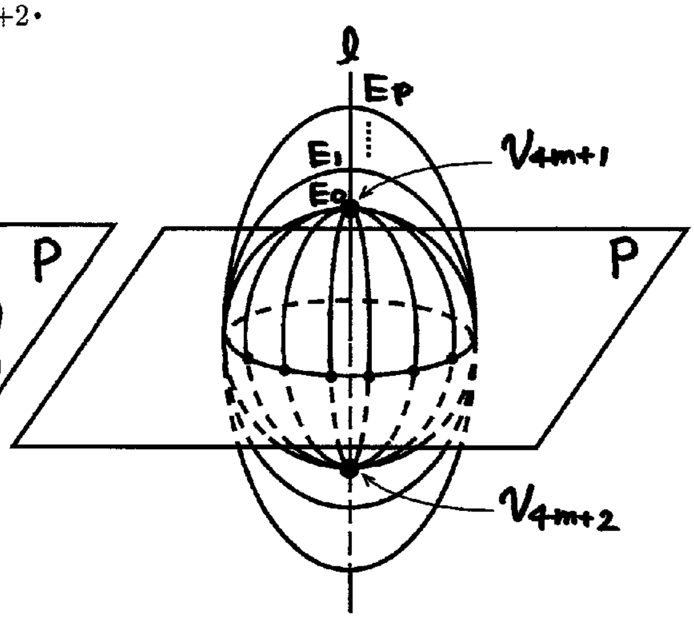

Fig. 6

§3. Sketch proof of Theorem 2

Theorem 2

In the case of $n \neq 4m+3$ (m is a non-negative integer), there exists an n-component achiral link with $\lambda_L \neq 0$.

Sketch proof)
Case 1 : construction of a $4m$-component achiral link

We consider a diagram of the achiral embedding of K_{4m} in §2. Fig.7 is a diagram of an achiral embedding of K_8.

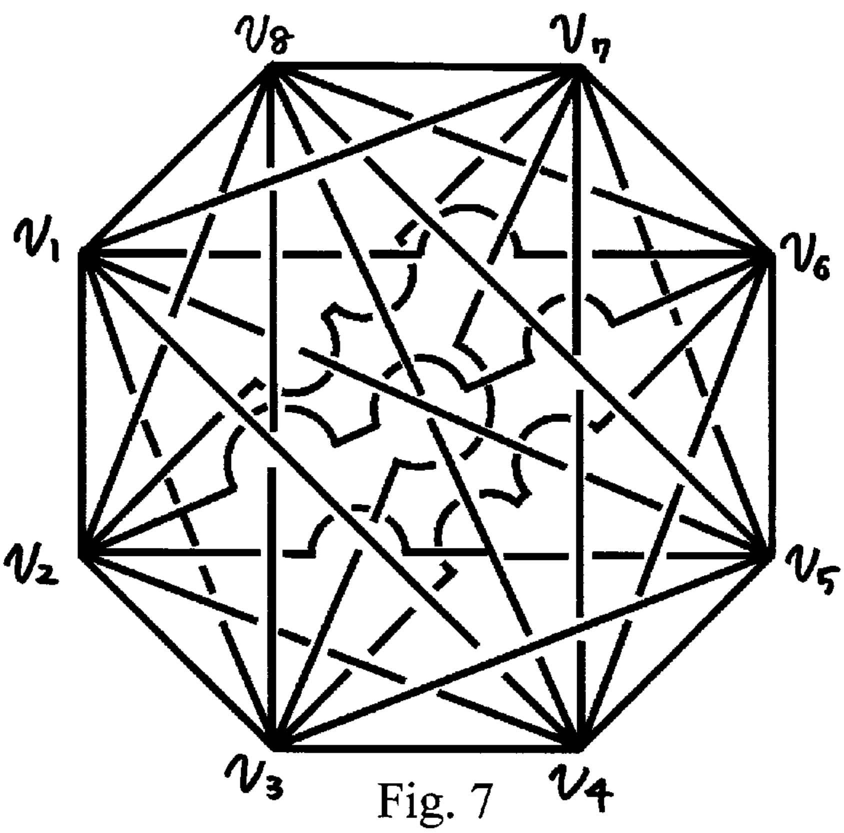

Fig. 7

We replace vertices with circles as shown in Fig.8.

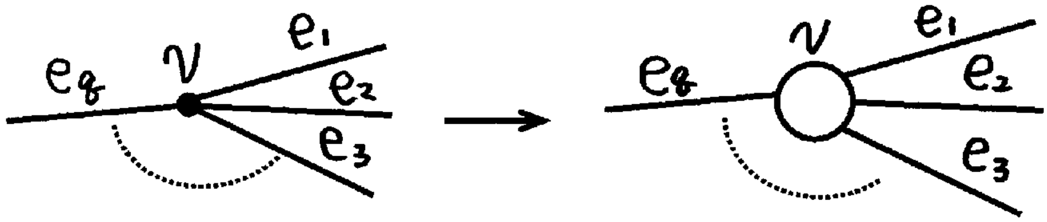

Fig. 8

Next, we replace edges with H_1 or H_2 as shown in Fig.9.

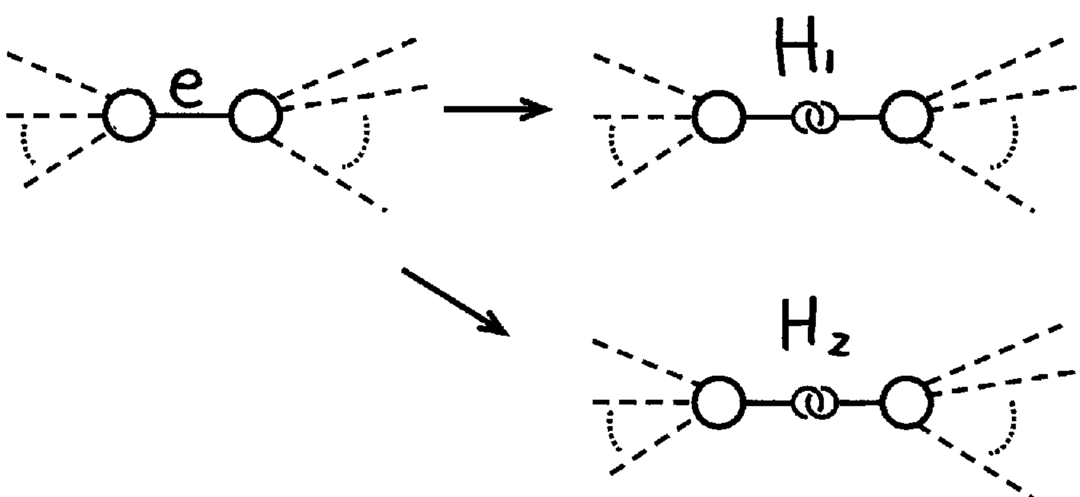

Fig. 9

By an edge $v_i v_{i+j}$ $(1 \leqq i \leqq m, 1 \leqq j \leqq 2m)$, we denote the representatives under α. Replace $v_i v_{i+j}$ with H_1, $\alpha(v_i v_{i+j})$ with H_2, $\alpha^2(v_i v_{i+j})$ with H_1, and $\alpha^3(v_i v_{i+j})$ with H_2. We have a diagram composed of circles and edges.

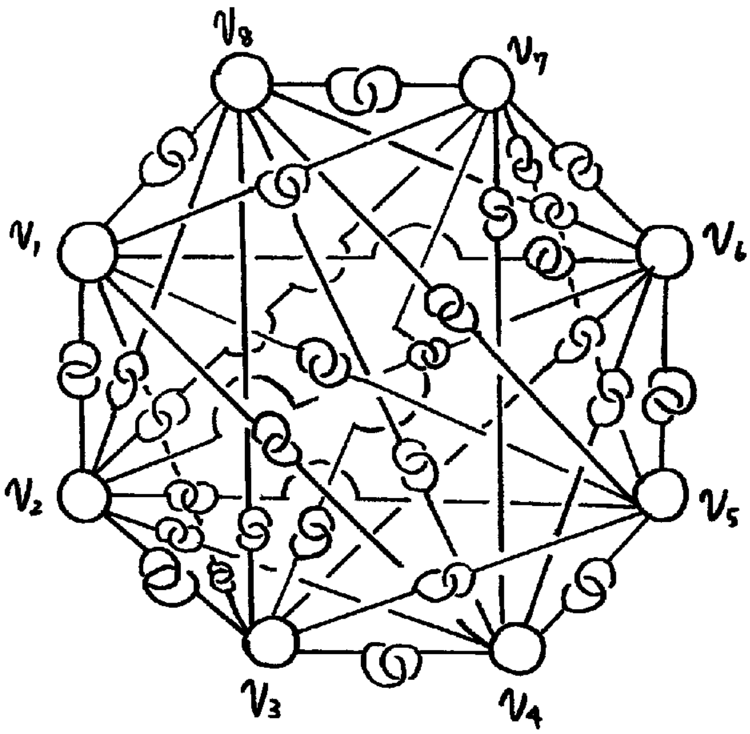

Fig. 10 (In the case of m=2)

We consider the operation on a diagram as shown in Fig.11 and call it a band sum along an edge.

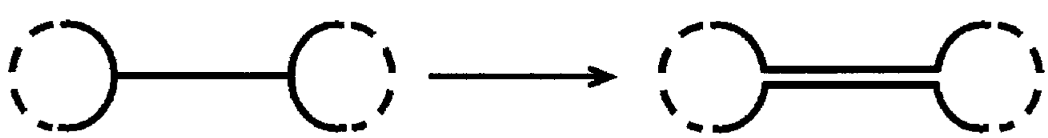

Fig. 11 Band sum along an edge

We operate the band sums along all edges connecting two circles. Then we have a link diagram.

Rotating the diagram by $90°$, we have the mirror image of it. That link is an example of a $4m$-component achiral link with $\lambda_L \neq 0$.

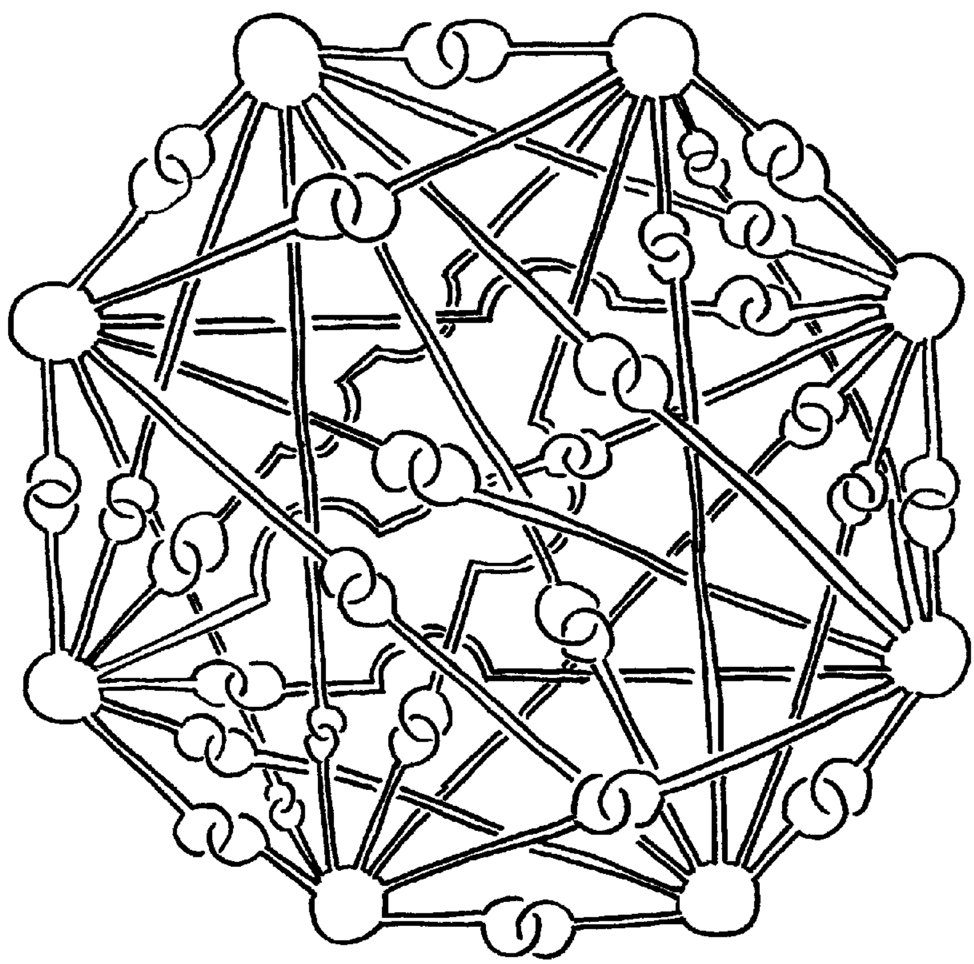

Fig. 12 8-component achiral link

Case 2 : construction of a $(4m+1)$-component achiral link

We consider a diagram of the achiral embedding of K_{4m+1} in §2.

From a vertex v_i $(i = 1, 2, \ldots, 4m)$, an edge $v_i v_{4m+1}$ goes out between edges $v_i v_{[i+2m]}$ and $v_i v_{[i+2m+1]}$, where $[k] = k$ if $k \leqq 4m$ and $[k] = k - 4m$ if $k > 4m$ for a positive number k.

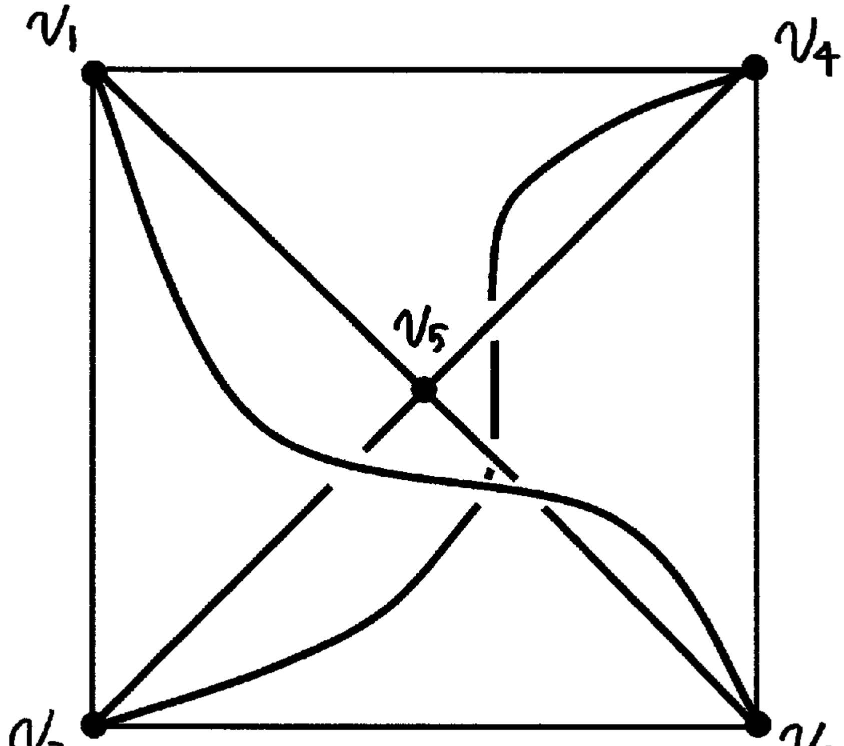

Fig. 13 (In the case of m=1)

We replace vertices with circles. And for $i = 1, 2, \cdots, m$, we replace edges $v_{4m+1}v_i$ and $\alpha^2(v_{4m+1}v_i)$ with H_1's, $\alpha(v_{4m+1}v_i)$ and $\alpha^3(v_{4m+1}v_i)$ with H_2's and the other edges as in Case 1.

We operate the band sums along all edges connected circles.

Then we have an achiral link diagram.

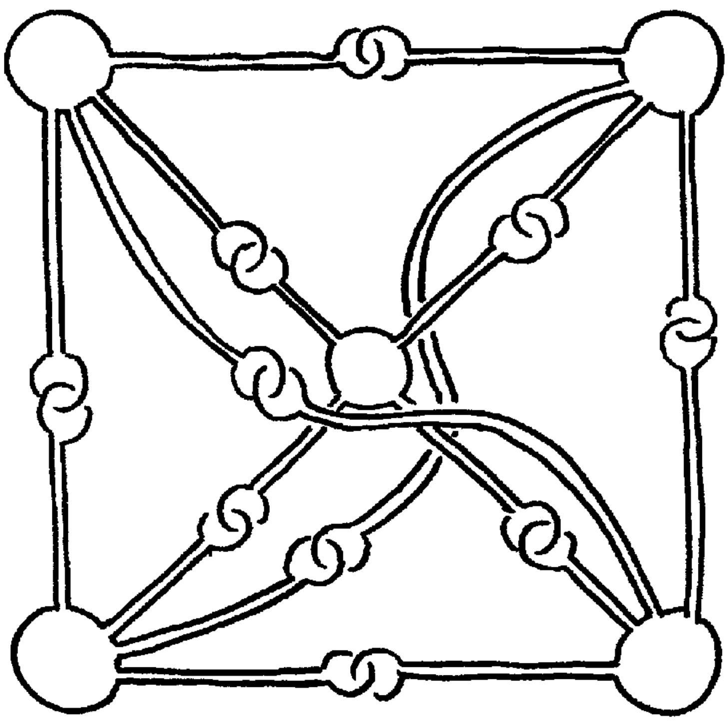

Fig. 14 5-component achiral link

Case 3: construction of a $(4m+2)$-component achiral link

We consider a diagram of the achiral embedding of K_{4m+2} in §2.

From a vertex v_i $(1 \leqq i \leqq 2m)$, edges $v_i v_{i+2m}$, $v_i v_{4m+2}$, $v_i v_{4m+1}$ and $v_i v_{[i+2m+1]}$ go out in this order in the counterclockwise direction.

From a vertex v_i $(2m+1 \leqq i \leqq 4m)$, edges $v_i v_{[i+2m]}$, $v_i v_{4m+1}$, $v_i v_{4m+2}$ and $v_i v_{[i+2m+1]}$ go out in this order in the counterclockwise direction.

We suppose that v_{4m+1} and v_{4m+2} are in an ε-neighborhood as shown in Fig.15.

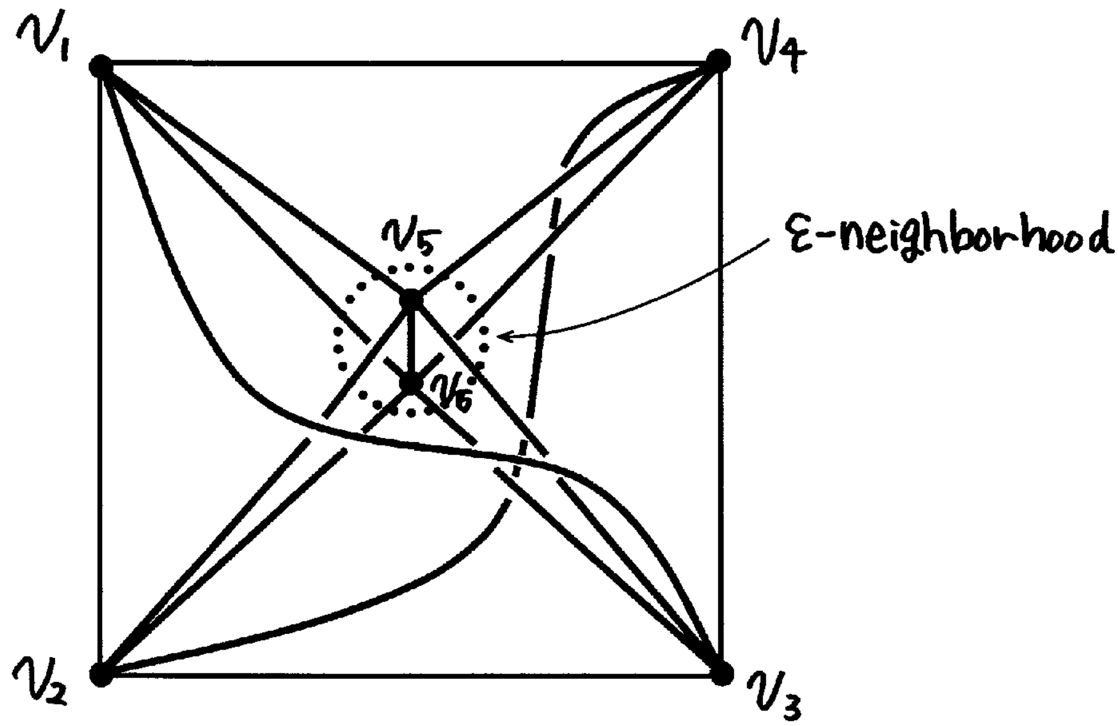

Fig. 15 (In the case of m=1)

First, we replace v_{4m+1} and v_{4m+2} with a Hopf link $C_{4m+1} \cup C_{4m+2}$.

We take $4m$ points p_i $(1 \leqq i \leqq 4m)$ on C_{4m+1} and $4m$ points q_i $(1 \leqq i \leqq 4m)$ on C_{4m+2} as shown in Fig.16.

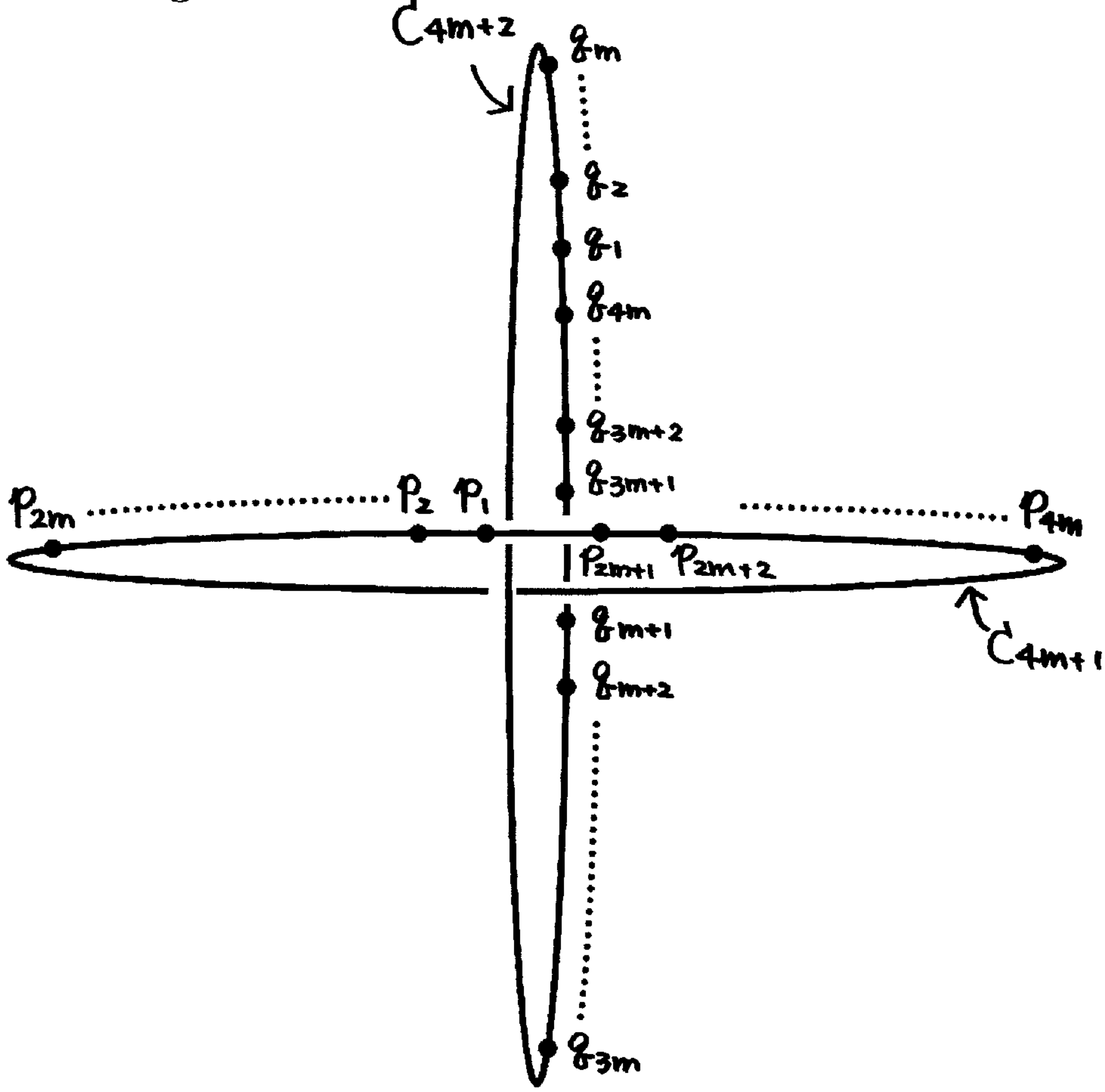

Fig. 16

We connect p_i to v_i and q_i to v_i by an edge as shown in Fig.17.

Fig.17 shows in the case of $m = 3$.

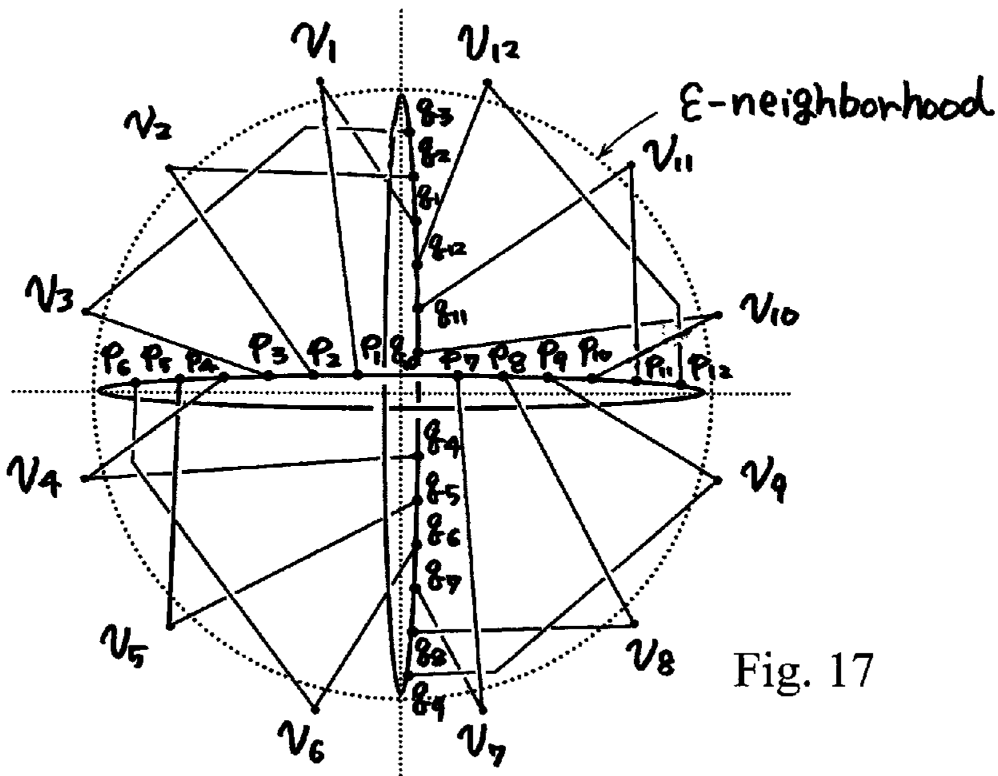

Fig. 17

An edge p_iv_i is under p_jv_j $(i > j)$ and over C_{4m+1}.
q_iv_i is over q_jv_j $(i > j)$ and under C_{4m+2}.
p_kv_k is over q_lv_l $(k, l = 1, 2, \cdots, 4m)$ in an ε-neighborhood.

Next, we replace a vertex v_i $(1 \leqq i \leqq 4m)$ with a circle as in Case 1. And we replace $p_i v_i$ with H_1 and $q_i v_i$ with H_2 $(i = 1, 2, \ldots, 4m)$. The other edges are replaced as in Case 1.

Finally, we operate the band sums along all edges connecting two circles. And we have a diagram of a $(4m+2)$-component link.

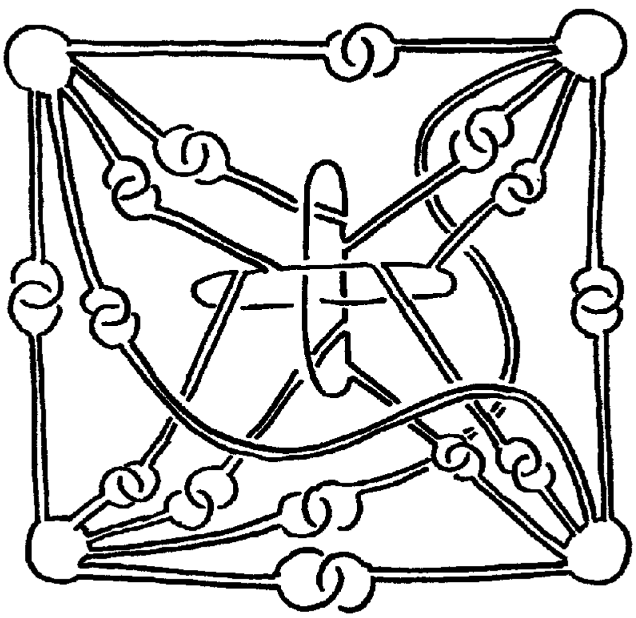

Fig. 18 6-component achiral link

Rotating the diagram by $90°$ and flyping a component of a Hopf link in an ε-neighborhood, we have a mirror image of the diagram.