Topological Symmetry Groups of K_1 to K_6 and K_{4r+3}

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The following definitions should be familiar by now.

Definition: $\mathrm{TSG}(\Gamma)$

The **topological symmetry group** of a graph Γ embedded in S^3 is the subgroup of $\mathrm{Aut}(\Gamma)$ induced by homeomorphisms of the graph in S^3. It is denoted by $\mathrm{TSG}(\Gamma)$

Definition: $\mathrm{TSG}_+(\Gamma)$

The **orientation preserving topological symmetry group**, $\mathrm{TSG}_+(\Gamma)$, is the subgroup of $\mathrm{TSG}(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

The Complete Graph on up to 3 vertices

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Thus K_3 is first exciting case.

Let Γ be an 'unknotted' embedding of K_3

Then $\mathrm{TSG}(\Gamma) = \mathrm{TSG}_+(\Gamma) = \mathrm{D}_3$

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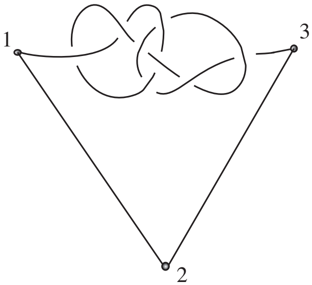

Figure: Knotted embedding of K_3.

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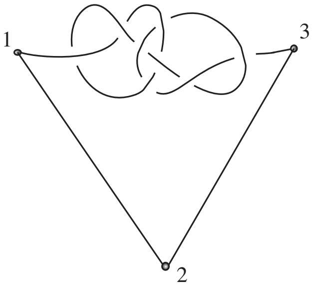

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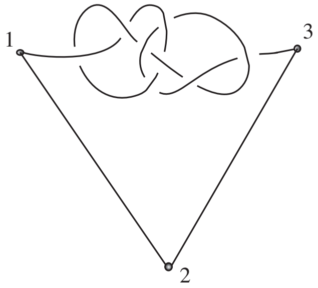

Figure: Knotted embedding of K_3.

8_{17} is non-invertible.

Thus $\mathrm{TSG}(\Gamma) = \mathrm{D}_3$ but $\mathrm{TSG}_+(\Gamma) = \mathbb{Z}_3$.

Note: Knots on K_3 can be slithered from edge to edge hence (123) is always an element in $\mathrm{TSG}(K_3)$ and thus we can't have $\mathrm{TSG}(K_3) = \mathbb{Z}_2$ or $\mathrm{TSG}(K_3) = \langle e \rangle$.

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Knots in Γ	$\mathrm{TSG}(\Gamma)$	$\mathrm{TSG}_+(\Gamma)$
None	D_3	D_3
8_{17}	D_3	$\mathbb{Z}_3$
$8_{17} \# 3_1$	$\mathbb{Z}_3$	$\mathbb{Z}_3$

Table: TSG Summary for K_3

In general, let Γ be a complete graph in S^3

The Complete Graph Theorem[2] is useful in paring down the possibilities for $\mathrm{TSG}_+(\Gamma)$.

Complete Graph Theorem (Flapan, Naimi and Tamvakis 2006)

A finite group H is $\mathrm{TSG}_+(\Gamma)$ for some embedding Γ of a complete graph in S^3 if and only if H is a finite cyclic group, a dihedral group, A_4, S_4 or A_5 or a subgroup of $\mathrm{D}_m \times \mathrm{D}_m$ for some odd m.

Other Key TSG_+ Theorems

Recall.

A_4 Theorem [Flapan, Mellor, Naimi]

A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that $\mathrm{TSG}_+(\Gamma) = A_4$ if and only if $m \equiv 0, 1, 4, 5, 8 \pmod{12}$.

S_4 Theorem [Flapan, Mellor, Naimi]

A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that $\mathrm{TSG}_+(\Gamma) = \mathrm{S}_4$ if and only if $m \equiv 0, 4, 8, 12, 20 \pmod{24}$.

A_5 Theorem [Flapan, Mellor, Naimi]

A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that $\mathrm{TSG}_+(\Gamma) = A_5$ if and only if $m \equiv 0, 1, 5, 20 \pmod{60}$.

Dihedral Theorem

A complete graph K_n has an embedding in S^3, Γ such that $\mathrm{TSG}_+(\Gamma) \cong \mathrm{D}_m$ if and only if one of the following holds:

(1) m is odd and $n = mr$, $mr + 1$, $mr + 2$, or $mr + 3$

(2) $m > 2$ is even and $m|n$

(3) m is 2 and n is even for $n > 2$

(4) m is 2 and $n = 4q + 1$ for some $q \in \mathbb{N}$

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Cyclic Theorem

A complete graph K_n has an embedding in S^3, Γ such that $\mathrm{TSG}_+(\Gamma) \cong \mathbb{Z}_m$ if and only if one of the following holds:

(1) m is odd and $n = mr$, $mr + 1$, $mr + 2$, or $mr + 3$

(2) $m > 2$ is even and $m|n$

(3) m is 2 and n is not 3

Auto-Order Lemma

Let $n > 3$ and let Γ be an embedding of K_n in S^3. Suppose that ϕ is an automorphism of Γ which is induced by an orientation preserving homeomorphism of (S^3, Γ). Then order$(\phi) \leq n$. Furthermore if ϕ is induced by an orientation reversing homeomorphism of (S^3, Γ) and $n = 5$ or 6 then order$(\phi) \leq 6$.

Chiral Knot Lemma

For any group G such that there exists an embedding Γ of a 3-connected graph in S^3 with $\mathrm{TSG}_+(\Gamma) \cong G$, then there also exists Γ' such that $\mathrm{TSG}(\Gamma') \cong G$.

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Theorems applied to first determine $\mathrm{TSG}_+(\Gamma)$ groups.

Subgroups of S_4.

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Table 2: Realizability of subgroups of S_4 as $\mathrm{TSG}_+(K_4)$

Subgroup	TSG_+	**Reason**
S_4	Yes	By S_4 Theorem
A_4	Yes	By A_4 Theorem
D_4	Yes	By Dihedral Theorem
D_3	Yes	By Dihedral Theorem
D_2	Yes	By Dihedral Theorem
$\mathbb{Z}_4$	Yes	By Cyclic Theorem
$\mathbb{Z}_3$	Yes	By Cyclic Theorem
$\mathbb{Z}_2$	Yes	By Cyclic Theorem

By Chiral Knot lemma above groups can also be $\mathrm{TSG}(K_4)$.

Subgroups of S_5

Subgroups of S_5

S_5, A_5, S_4, A_4, $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, D_6, D_5, D_4, D_3, D_2, $\mathbb{Z}_6$, $\mathbb{Z}_5$, $\mathbb{Z}_4$, $\mathbb{Z}_3$, $\mathbb{Z}_2$

$A \rtimes B$ - semidirect product of a group B acting on a group A.

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$A \rtimes B$ - semidirect product of a group B acting on a group A.

By the Complete Graph Theorem we can't have $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ or S_5 as $\mathrm{TSG}_+(K_5)$.

A_5, S_4, A_4, D_6, D_5, D_4, D_3, D_2, $\mathbb{Z}_6$, $\mathbb{Z}_5$, $\mathbb{Z}_4$, $\mathbb{Z}_3$, and $\mathbb{Z}_2$ are the only candidates for $\mathrm{TSG}_+(K_5)$

$\mathrm{TSG}_+(K_5)$ results

Table 3: Realizability of subgroups of S_5 as $\mathrm{TSG}_+(K_5)$

Subgroup	TSG_+	**Reason**
A_5	Yes	By A_5 Theorem
S_4	No	By S_4 Theorem
A_4	Yes	By A_4 Theorem
D_6	No	By Auto-Order Lemma
D_5	Yes	By Dihedral Theorem
D_4	No	By Dihedral Theorem
D_3	Yes	By Dihedral Theorem
D_2	Yes	By Dihedral Theorem
$\mathbb{Z}_6$	No	By Auto-Order Lemma
$\mathbb{Z}_5$	Yes	By Cyclic Theorem
$\mathbb{Z}_4$	No	By Cyclic Theorem
$\mathbb{Z}_3$	Yes	By Cyclic Theorem
$\mathbb{Z}_2$	Yes	By Cyclic Theorem

By Chiral Knot lemma each group that was “yes” in Table 3 can also be $\mathrm{TSG}(K_5)$.

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Need to consider S_5, S_4, D_6, D_4, $\mathbb{Z}_6$, $\mathbb{Z}_4$ and $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ for $\mathrm{TSG}(K_5)$.

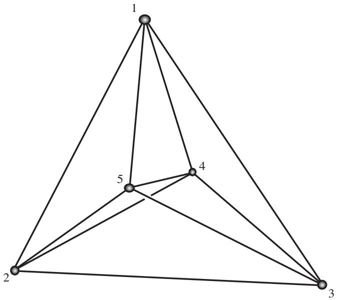

Figure: $\mathrm{TSG}(\Gamma) = \mathrm{S}_5$.

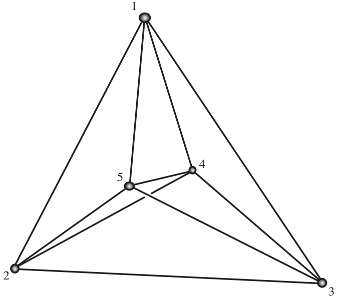

Figure: $\mathrm{TSG}(\Gamma) = \mathrm{S}_5$.

Thus $\mathrm{TSG}(K_5)$ can be S_5 and S_4.

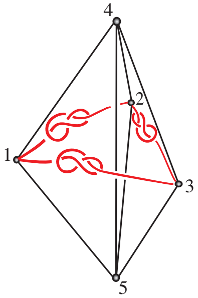

Figure: $\mathrm{TSG}(\Gamma) = \mathrm{D}_6$.

$\mathrm{TSG}(K_5)$ results contd...

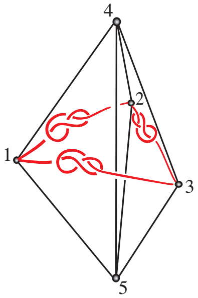

Figure: $\mathrm{TSG}(\Gamma) = \mathrm{D}_6$.

Automorphism (123)(45) has order 6.

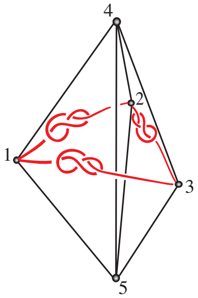

Figure: $\mathrm{TSG}(\Gamma) = \mathrm{D}_6$.

Automorphism (123)(45) has order 6.

Thus $\mathrm{TSG}(K_5) = \langle (123), (45), (23) \rangle = \mathrm{D}_6$.

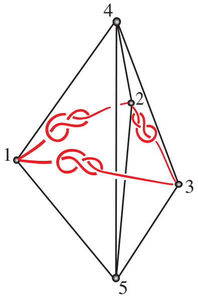

Figure: $\mathrm{TSG}(\Gamma) = \mathrm{D}_6$.

Automorphism (123)(45) has order 6.

Thus $\mathrm{TSG}(K_5) = \langle (123), (45), (23) \rangle = \mathrm{D}_6$.

Add 8_{17} to red edges to obtain $\mathrm{TSG}(K_5) = \mathbb{Z}_6$

$\mathrm{TSG}(K_5)$ results contd...

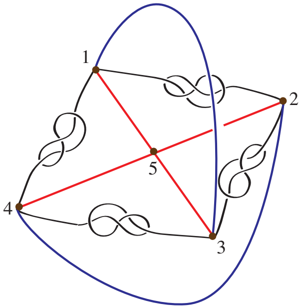

Figure: $\mathrm{TSG}(\Gamma) = \mathrm{D}_4$.

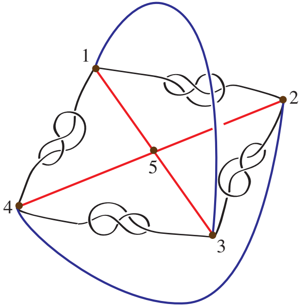

Figure: $\mathrm{TSG}(\Gamma) = \mathrm{D}_4$.

Thus $\mathrm{TSG}(K_5)$ can be D_4 and $\mathbb{Z}_4$.

Table 4: Realizability of subgroups of S_5 as $\mathrm{TSG}(K_5)$

Subgroup	TSG	**Reason**
S_5	Yes	See above sketch and argument
A_5	Yes	By Chiral Knot Lemma
S_4	Yes	See above sketch and argument
A_4	Yes	By Chiral Knot Lemma
D_6	Yes	See above sketch and argument
D_5	Yes	By Chiral Knot Lemma
D_4	Yes	See above sketch and argument
D_3	Yes	By Chiral Knot Lemma
D_2	Yes	By Chiral Knot Lemma
$\mathbb{Z}_6$	Yes	See above sketch and argument
$\mathbb{Z}_5$	Yes	By Chiral Knot Lemma
$\mathbb{Z}_4$	Yes	See above sketch and argument
$\mathbb{Z}_3$	Yes	By Chiral Knot Lemma
$\mathbb{Z}_2$	Yes	By Chiral Knot Lemma

Last $\mathrm{TSG}(K_5)$ result

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All other results have been "Yes" so far...

Subgroups of S_6:

S_6, A_6, S_5, A_5, $\mathrm{S}_3 \wr \mathbb{Z}_2$, $\mathrm{S}_4 \times \mathbb{Z}_2$, $A_4 \times \mathbb{Z}_2$, S_4, A_4, $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, $\mathrm{D}_3 \times \mathrm{D}_3$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, $\mathrm{D}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, D_6, D_5, D_4, $\mathrm{D}_4 \times \mathbb{Z}_2$, D_3, D_2, $\mathbb{Z}_6$, $\mathbb{Z}_5$, $\mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Note $A \wr B$ represents a wreath product of A by B.

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S_6, A_6, S_5, A_5, $\mathrm{S}_3 \wr \mathbb{Z}_2$, $\mathrm{S}_4 \times \mathbb{Z}_2$, $A_4 \times \mathbb{Z}_2$, S_4, A_4, $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, $\mathrm{D}_3 \times \mathrm{D}_3$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, $\mathrm{D}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, D_6, D_5, D_4, $\mathrm{D}_4 \times \mathbb{Z}_2$, D_3, D_2, $\mathbb{Z}_6$, $\mathbb{Z}_5$, $\mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

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Approach will be similar but a lot more groups to consider.

And now onto $\mathrm{TSG}_+(K_{4r+3})$

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Why K_{4r+3}?

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The Complete Graph theorem tells us $\mathrm{TSG}_+(\Gamma)$ is a subgroup of S_{15} which is either cyclic, dihedral, A_4, S_4 or A_5 or a subgroup of $\mathrm{D}_m \times \mathrm{D}_m$.

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That's a lot less work than sorting through 1,307,674,368,000 elements of S_{15} looking for possible subgroups.

Complete but...

Complete Graph Theorem significantly reduces the number of possible groups that can be $\mathrm{TSG}_+(\Gamma)$ for a complete graph, Γ...

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...but there are still quite a few possibilities left to check. In particular ALL the subgroups of $\mathrm{D}_m \times \mathrm{D}_m$.

The beginning of a new theorem

Lemma (Flapan 1995)

If ϕ is an order 2 automorphism induced by an orientation preserving homeomorphism of K_n in S^3, then all cycles of ϕ are of order 2 and there are at most 2 fixed vertices.

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We use the above lemma to prove the following key lemma.

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If ϕ is an order 2 automorphism induced by an orientation preserving homeomorphism of K_n in S^3, then all cycles of ϕ are of order 2 and there are at most 2 fixed vertices.

We use the above lemma to prove the following key lemma.

No D_2 Lemma

There is no embedding Γ of K_{4r+3} in S^3 such that $\mathrm{D}_2 \leq \mathrm{TSG}_+(\Gamma)$.

Why is the No D_2 Lemma significant?

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A_4, S_4, A_5, and $\mathrm{D}_m \times \mathrm{D}_m$ all have D_2 as a subgroup.

Implications of No D_2

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A_4, S_4, A_5, and $\mathrm{D}_m \times \mathrm{D}_m$ all have D_2 as a subgroup.

Proving the No D_2 Lemma significantly reduces the possibilities from the Complete Graph Theorem.

Summary of Proof Story

In our research we were able to prove a number of lemmas to greatly reduce the number of possibilities for $\mathrm{TSG}_+(K_{4r+3})$. Thus obtaining the forward direction of our theorem which determines which groups are "candidates" for $\mathrm{TSG}_+(K_{4r+3})$.

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We then employed tailored versions of the Edge Embedding Lemma[3], the Subgroups Theorem[4] and some creative sketches to prove that each of the above groups ("candidates") actually can occur as $\mathrm{TSG}_+(\Gamma)$ for some embedding of K_{4r+3}.

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We then employed tailored versions of the Edge Embedding Lemma[3], the Subgroups Theorem[4] and some creative sketches to prove that each of the above groups ("candidates") actually can occur as $\mathrm{TSG}_+(\Gamma)$ for some embedding of K_{4r+3}.

Leading to the following "if and only if" Theorem....

A complete characterization of exactly which groups occur as $\mathrm{TSG}_+(\Gamma)$ for some embedding Γ of K_{4r+3}.

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Our Theorem

Let $n = 4r + 3$. A finite group G is $\mathrm{TSG}_+(\Gamma)$ for some embedding Γ of K_n in S^3 if and only if one of the following holds:

1. G is $\mathbb{Z}_2$
2. G is D_p or $\mathbb{Z}_p$ for p odd and $p|n$, $p|n-1$, $p|n-2$, or $p|n-3$.
3. $G = \mathbb{Z}_p \times \mathbb{Z}_q$ where p and q are odd, $p|q$ and
   - If $p > 3$, then $pq|n$.
   - If $p = 3$, $q \neq 3$, then $pq|n$ or $pq|n-3$.
   - If $p = q = 3$, then $pq|n$ or $pq|n-3$ or $pq|n-6$.

Let's apply our theorem to K_{15}....

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Our Theorem for n=15

Let Γ be an embedding of K_{15} in S^3 such that $G = \mathrm{TSG}_+(\Gamma)$. Then one of the following holds:

1. G is $\mathbb{Z}_2$

Let's apply our theorem to K_{15}....

Our Theorem for n=15

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1. G is $\mathbb{Z}_2$

So $\mathbb{Z}_2$ is one of the groups.

Our Theorem contd...

2. G is D_p or $\mathbb{Z}_p$ for p odd and $p|15$, $p|14$, $p|13$, or $p|12$.

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Task 2: Find all odd numbers which meet criteria 2.

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So $\mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_7, \mathbb{Z}_{13}, \mathbb{Z}_{15}$ are among the groups.

Our Theorem contd...

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As well as $\mathrm{D}_3, \mathrm{D}_5, \mathrm{D}_7, \mathrm{D}_{13}, \mathrm{D}_{15}$.

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As well as $\mathrm{D}_3, \mathrm{D}_5, \mathrm{D}_7, \mathrm{D}_{13}, \mathrm{D}_{15}$.

These are the only groups which meet criteria 2.

Our Theorem contd...

3. $G = \mathbb{Z}_p \times \mathbb{Z}_q$ where p and q are odd, $p|q$ and
   (a) If $p > 3$, then $pq|15$.
   (b) If $p = 3$, $q \neq 3$, then $pq|15$ or $pq|12$.
   (c) If $p = q = 3$, then $pq|15$ or $pq|12$ or $pq|9$.

Our Theorem contd...

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   (b) If $p = 3$, $q \neq 3$, then $pq|15$ or $pq|12$.
   (c) If $p = q = 3$, then $pq|15$ or $pq|12$ or $pq|9$.

Task 3: Find p and q as defined above.

Our Theorem contd...

3. $G = \mathbb{Z}_p \times \mathbb{Z}_q$ where p and q are odd, $p|q$ and
   (a) If $p > 3$, then $pq|15$.
   (b) If $p = 3$, $q \neq 3$, then $pq|15$ or $pq|12$.
   (c) If $p = q = 3$, then $pq|15$ or $pq|12$ or $pq|9$.

Task 3: Find p and q as defined above.

(a) Not possible

Our Theorem contd...

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Task 3: Find p and q as defined above.

(a) Not possible

(b) Not possible

Our Theorem contd...

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Task 3: Find p and q as defined above.

(a) Not possible

(b) Not possible

(c) $\mathbb{Z}_3 \times \mathbb{Z}_3$ is the only group which satisfies this condition.

Our theorem tells us that for Γ an embedding of K_{15} in S^3, $G = \mathrm{TSG}_+(\Gamma)$ must be one of the following:

Good results

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$$\mathbb{Z}_2$$

$$\mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_7, \mathbb{Z}_{13}, \mathbb{Z}_{15}$$

$$\mathrm{D}_3, \mathrm{D}_5, \mathrm{D}_7, \mathrm{D}_{13}, \mathrm{D}_{15}$$

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An exact list.

Thanks

While the study of $\mathrm{TSG}(K_n)$ for $n < 7$ is my independent research, the study of $\mathrm{TSG}_+(K_{4r+3})$ was first attempted by John D. O'Brien in his 2002 Pomona Senior Mathematics thesis.

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Thanks to John D. O'Brien, Michael Yoshizawa and Professor Flapan.

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