Background
○○○○

Results
○

Necessity of the Conditions
○○○○○

Sufficiency of the Conditions
○○○○○○○

Further Questions
○○

Topological Symmetry Groups of Complete Graphs

Blake Mellor

Loyola Marymount University

International Workshop on Spatial Graphs, 2010
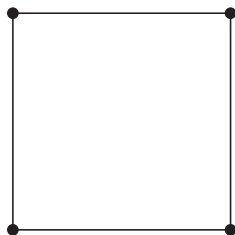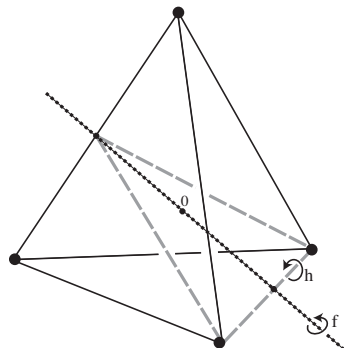
Topological Symmetry Groups

The *topological symmetry group* was introduced by Jon Simon (1986) to study the symmetries of molecules whose structure is not rigid. It can be used to study the symmetries of any graph embedded in S^3.

Definition

Let γ be an abstract graph with automorphism group $\mathrm{Aut}(\gamma)$. Let Γ be an embedding of γ in S^3. The (orientation preserving) topological symmetry group of Γ, denoted $\mathrm{TSG}_+(\Gamma)$, is the subgroup of $\mathrm{Aut}(\gamma)$ induced by orientation preserving homeomorphisms of the pair (S^3, Γ).

Examples

Below are embeddings Γ_1 of $K_{2,2}$ and Γ_2 of K_4 where $\mathrm{TSG}_+(\Gamma_1) = D_4$ and $\mathrm{TSG}_+(\Gamma_2) = S_4$ (with involutions f and h).

Γ_1

Γ_2

Complete Graph Theorem

Which groups can be topological symmetry groups for a complete graph?

Complete Graph Theorem (Flapan, Naimi, Tamvakis)

A finite group H is $\mathrm{TSG}_+(\Gamma)$ *for some embedding* Γ *of a complete graph in* S^3 *if and only if H is isomorphic to a finite subgroup of either* $\mathrm{SO}(3)$ *or* $D_m \times D_m$ *for some odd m.*

The finite subgroups of $\mathrm{SO}(3)$ are the cyclic and dihedral groups, and the *polyhedral groups* A_4, S_4 and A_5.

The Complete Graph Theorem does *not* specify which complete graphs can realize a particular group *H*.

Our Question

Question

For which m does K_m have an embedding Γ such that $\mathrm{TSG}_+(\Gamma)$ is isomorphic to A_4, S_4 or A_5?

Recall that A_4 is the group of symmetries of a solid tetrahedron, S_4 is the group of symmetries of a solid cube (or of a hollow tetrahedron), and A_5 is the group of symmetries of a solid dodecahedron (or of a 4-simplex).

Background	**Results**	Necessity of the Conditions	Sufficiency of the Conditions	Further Questions
oooo	●	ooooo	ooooooo	oo

Main Results

A_4 Theorem

A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that $\mathrm{TSG}_+(\Gamma) = A_4$ if and only if $m \equiv 0, 1, 4, 5, 8 \pmod{12}$.

S_4 Theorem

A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that $\mathrm{TSG}_+(\Gamma) = S_4$ if and only if $m \equiv 0, 4, 8, 12, 20 \pmod{24}$.

A_5 Theorem

A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that $\mathrm{TSG}_+(\Gamma) = A_5$ if and only if $m \equiv 0, 1, 5, 20 \pmod{60}$.

Proving Necessity

1. We first observe the number of vertices fixed by an element of the symmetry group is determined by the conjugacy class of the element. In particular, in A_4 and A_5, two elements of the same order must fix the same number of vertices (in S_4 there are two conjugacy classes of elements of order 2).

   The necessity of the conditions rests on determining the possible combinations of the number of fixed vertices for each conjugacy class.

Proving Necessity

2 We next recall the following result of Flapan, Naimi and Tamvakis:

Isometry Theorem (Flapan, Naimi, Tamvakis)

Let Ω be an embedding of K_m in S^3 such that $\mathrm{TSG}_+(\Omega)$ is not a cyclic group of odd order. Then K_m can be re-embedded in S^3 as Γ such that $\mathrm{TSG}_+(\Omega) \leq \mathrm{TSG}_+(\Gamma)$ and $\mathrm{TSG}_+(\Gamma)$ is induced by an isomorphic finite subgroup of $\mathrm{SO}(4)$.

As a result, we may assume that all symmetries are either rotations (which fix a circle) or glide rotations (with no fixed points). This means that no symmetry can fix more than 3 vertices, since K_4 cannot be embedded in a circle.

Proving Necessity

3. We now have a fairly small number of possible combinations for the number of vertices fixed by each element of the group; careful analysis allows us to reduce the combinations even further. For example, no involution in A_4 or A_5 can fix more than one vertex.

   This leaves only a few possibilities. For example, if $A_4 \leq \mathrm{TSG}_+(\Gamma)$, let n_2 be the number of vertices fixed by the elements of order 2, and n_3 be the number of vertices fixed by the elements of order 3. Then the possibilities for (n_2, n_3) are just $(0, 0)$, $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 1)$ and $(1, 2)$.

Proving Necessity

4 Finally, we take these remaining possibilities and apply Burnside's Lemma:

Burnside's Lemma

Suppose a group G acts on the vertices of an embedded graph Γ. Let $\operatorname{fix}(g)$ be the set of vertices fixed by $g \in G$. Then the number of vertex orbits is given by:

$$\# \text{ orbits} = \frac{1}{|G|} \sum_{g \in G} |\operatorname{fix}(g)|$$

Knowing the number of orbits must be an integer, and that the identity fixes all the vertices, allows us to determine the possible number of vertices in the complete graph, modulo the size of the group.

Example: Necessity for A_4

Recall that $|A_4| = 12$, and consider the graph K_m. For each combination of n_2 and n_3, the value of $m \pmod{12}$ is determined by knowing that $\frac{1}{12}(m + 3n_2 + 8n_3)$ is an integer (the identity fixes m vertices).

n_2	n_3	$m \pmod{12}$
0	0 or 3	0
0	1	4
0	2	8
1	1	1
1	2	5

Proving Sufficiency

To prove the sufficiency of the conditions, we will construct embeddings of the complete graphs with the desired topological symmetry groups. For a given group G, this is done by:

1. Embedding the vertices of the graph so there is a faithful action of G.
2. Showing that we can extend the embedding to the edges so that $G \leq \mathrm{TSG}_+(\Gamma)$
3. Modifying the embedding to restrict the symmetries to G.

We embed the vertices using the regular polyhedra. The second step, embedding the edges, requires an Edge Embedding Lemma.

The Edge Embedding Lemma

Edge Embedding Lemma

Let G be a finite subgroup of $\mathrm{Diff}_+(S^3)$ which acts faithfully on the embedded vertices V of a graph γ. Suppose that adjacent pairs of vertices in V satisfy the following hypotheses:

1 *If a pair is pointwise fixed by non-trivial elements $h, g \in G$, then $\mathrm{fix}(h) = \mathrm{fix}(g)$.*

2 *For each pair $\{v, w\}$ in the fixed point set C of some non-trivial element of G, there is an arc $A_{vw} \subseteq C$ bounded by $\{v, w\}$ whose interior is disjoint from V and from any other such arc $A_{v'w'}$.*

3 *If a point in the interior of some A_{vw} or a pair $\{v, w\}$ bounding some A_{vw} is setwise invariant under an $f \in G$, then $f(A_{vw}) = A_{vw}$.*

4 *If a pair is interchanged by some $g \in G$, then the subgraph of γ whose vertices are pointwise fixed by g can be embedded in a proper subset of a circle.*

5 *If a pair is interchanged by some $g \in G$, then $\mathrm{fix}(g)$ is non-empty, and for any $h \neq g$, then $\mathrm{fix}(h) \neq \mathrm{fix}(g)$.*

Then there is an embedding of the edges of γ in S^3 such that the resulting embedding of γ is setwise invariant under G.

Using the Edge Embedding Lemma

The Edge Embedding Lemma allows us to embed a graph so that it has certain symmetries (and possibly others as well) by embedding the vertices so that certain conditions are satisfied. This is much easier than trying to embed all the edges by hand!

For complete graphs, if the desired group is S_4 or A_5, the Complete Graph Theorem says that these cannot be proper subgroups of the topological symmetry group, so we're done. If the desired group is A_4, we may need to modify the embedding to ensure the symmetry group is not S_4 or A_5.

Example: Embedding K_{24n+20} so that $\mathrm{TSG}_+ = S_4$
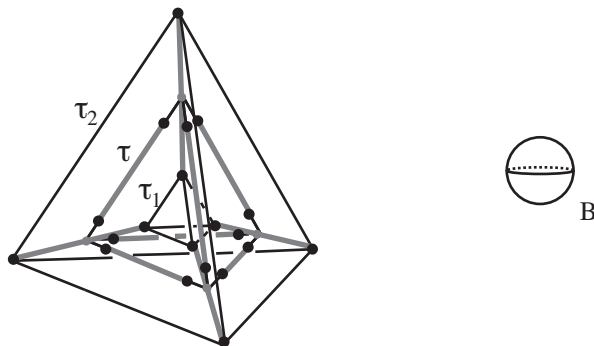
The gray arcs are the arcs required by the Edge Embedding Lemma. The involutions not in A_4 interchange τ_1 and τ_2.

Subgroup Theorem

We apply the Subgroup Theorem to our embeddings for S_4 and A_5 to obtain embeddings for A_4.

Subgroup Theorem

*Let Γ be an embedding of a 3-connected graph in S^3. Suppose that Γ contains an edge e which is not pointwise fixed by any non-trivial element of $\mathrm{TSG}_+(\Gamma)$. Then for every $H \leq \mathrm{TSG}_+(\Gamma)$, there is an embedding Γ' of Γ with $H = \mathrm{TSG}_+(\Gamma')$.*

For example, in the embedding for K_{20}, any of the edges not marked in gray will satisfy the hypothesis.
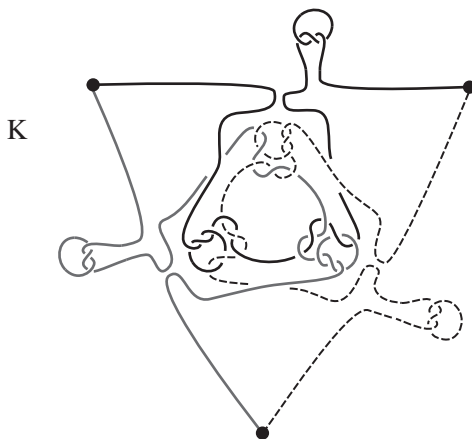
Special Case: An Embedding of K_4 with $\mathrm{TSG}_+ = A_4$

Embed the vertices of K_4 as the vertices of the standard tetrahedron, and embed the edges as shown (when the tetrahedron is unfolded):
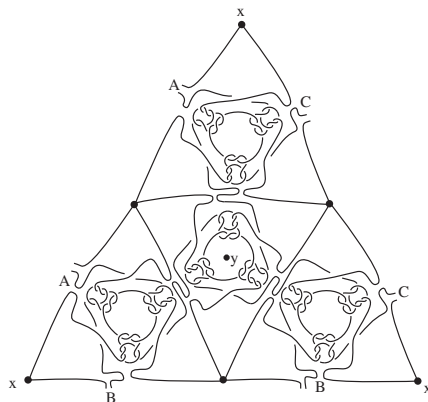
Special Case: An Embedding of K_4 with $\mathrm{TSG}_+ = A_4$

Then the three edges around each face form the following non-invertible knot:

Further Questions

- If G is a cyclic or dihedral group, or a subgroup of $D_m \times D_m$ with m odd, for which n does K_n have an embedding Γ such that $\mathrm{TSG}_+(\Gamma) = G$? (Work in progress.)

- In general, if γ is a 3-connected graph, the topological symmetry group can only be a subgroup of $\mathrm{SO}(4)$ (Flapan, Naimi, Pommersheim, Tamvakis). For any such subgroup, can we characterize the graphs which can realize that group as a topological symmetry group?

Thank You

- Thank you all for coming to this talk.
- Thanks to the organizers and Waseda University for hosting this conference!
- Preprint is available at *arXiv:1008.1095*
- Any questions?