Topological Symmetry Groups and Local Knotting

Erica Flapan

August 18, 2010

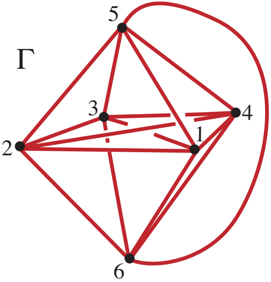
International Workshop on Spatial Graphs
Waseda University

Studying the symmetries of graphs in S^3 is a natural extension of studying symmetries of knots.

A symmetry of a graph in S^3 induces an automorphism of the vertices which can represent the symmetry.

Studying the symmetries of graphs in S^3 is a natural extension of studying symmetries of knots.

A symmetry of a graph in S^3 induces an automorphism of the vertices which can represent the symmetry.



A rotation by 90° followed by a reflection induces the automorphism $(1234)(56)$.

Definition

An automorphism φ of an abstract graph γ is **realizable** if there is some embedding Γ of γ in S^3 such that a homeomorphism h of (S^3, Γ) induces φ on Γ.

We saw that $(1234)(56)$ is a realizable automorphism of K_6.

Definition

An automorphism φ of an abstract graph γ is **realizable** if there is some embedding Γ of γ in S^3 such that a homeomorphism h of (S^3, Γ) induces φ on Γ.

We saw that (1234)(56) is a realizable automorphism of K_6.

Theorem [F]

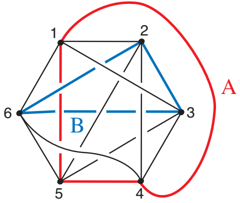
The automorphism (1234) of K_6 is not realizable.

To prove this, we use the following.

Definition

For an embedding Γ of K_6 in S^3, let Ω be the mod 2 sum of the linking numbers of all triangle pairs in Γ:

$$\Omega(\Gamma) = \sum_{A,B \subseteq \Gamma} \mathrm{lk}(A,B) \pmod 2$$

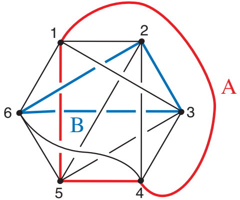


lk(A,B)=1

Definition

For an embedding Γ of K_6 in S^3, let Ω be the mod 2 sum of the linking numbers of all triangle pairs in Γ:

$$\Omega(\Gamma) = \sum_{A,B \subseteq \Gamma} \mathrm{lk}(A, B) \pmod 2$$



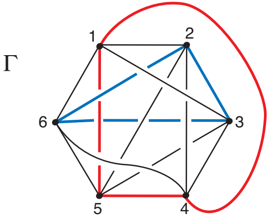
lk(A,B)=1

Theorem [Conway and Gordon]

For any embedding Γ of K_6 in S^3, $\Omega(\Gamma) = 1$.

Suppose (1234) is induced by a homeomorphism h of some embedding Γ of K_6 in S^3.

3 vertices in Γ define a pair of disjoint triangles and hence their linking number.



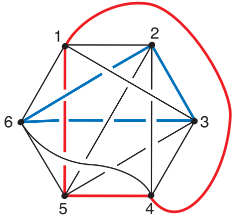
415 determines this pair

Proof that (1234) is not realizable

Suppose (1234) is induced by a homeomorphism h of some embedding Γ of K_6 in S^3.

3 vertices in Γ define a pair of disjoint triangles and hence their linking number.

Γ



415 determines this pair

The orbits of all 10 pairs of disjoint triangles in Γ under (1234) are:

$< 123, 234, 341, 412 >$
$< 125, 235, 345, 415 >$
$< 135, 245 >$

Proof that (1234) is not realizable

Since h induces (1234), the linked triangles in a given orbit all have the same (mod 2) linking number.

Since h induces (1234), the linked triangles in a given orbit all have the same (mod 2) linking number.

Since each orbit has an even number of elements,

$$\Omega(\Gamma) = \sum_{A,B \subseteq \Gamma} \mathrm{lk}(A, B) \equiv 0 \pmod 2$$

Since h induces (1234), the linked triangles in a given orbit all have the same (mod 2) linking number.

Since each orbit has an even number of elements,

$$\Omega(\Gamma) = \sum_{A,B \subseteq \Gamma} \mathrm{lk}(A,B) \equiv 0 \pmod 2$$

$\Longrightarrow \Longleftarrow$ to Conway and Gordon. So (1234) is not realizable. ∎

Proof that (1234) is not realizable

Since h induces (1234), the linked triangles in a given orbit all have the same (mod 2) linking number.

Since each orbit has an even number of elements,

$$\Omega(\Gamma) = \sum_{A,B \subseteq \Gamma} \mathrm{lk}(A,B) \equiv 0 \pmod 2$$

$\Longrightarrow \Longleftarrow$ to Conway and Gordon. So (1234) is not realizable. $\blacksquare$

Definition

The **topological symmetry group** of a graph Γ embedded in S^3 $\mathrm{TSG}(\Gamma)$ is the subgroup of the automorphism group, $\mathrm{Aut}(\Gamma)$, induced by homeomorphisms of (S^3, Γ).

Since h induces (1234), the linked triangles in a given orbit all have the same (mod 2) linking number.

Since each orbit has an even number of elements,

$$\Omega(\Gamma) = \sum_{A,B \subseteq \Gamma} \mathrm{lk}(A,B) \equiv 0 \pmod 2$$

$\Longrightarrow \Longleftarrow$ to Conway and Gordon. So (1234) is not realizable. $\blacksquare$

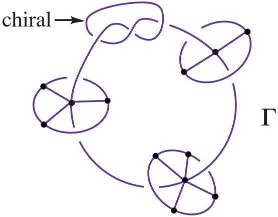
Definition

The **topological symmetry group** of a graph Γ embedded in S^3 $\mathrm{TSG}(\Gamma)$ is the subgroup of the automorphism group, $\mathrm{Aut}(\Gamma)$, induced by homeomorphisms of (S^3, Γ).

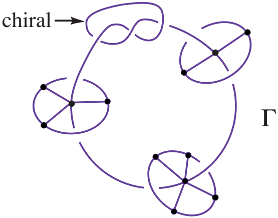
By the above Theorem, for all $n \geq 6$, $\mathrm{TSG}(K_n) \neq S_n$.

What groups can be $\mathrm{TSG}(\Gamma)$ for some Γ?

What groups can be $\mathrm{TSG}(\Gamma)$ for some Γ?



Wheels can rotate but can't be interchanged. Because of chiral knot, there are no orientation reversing homeomorphisms.



Wheels can rotate but can't be interchanged. Because of chiral knot, there are no orientation reversing homeomorphisms.

$$\mathrm{TSG}(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$$

Any finite abelian group can be $\mathrm{TSG}(\Gamma)$

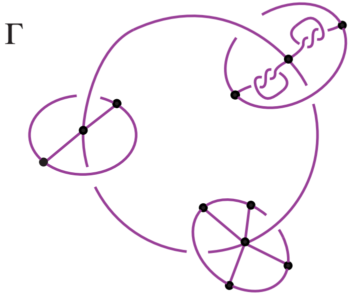
We can have any number of wheels with any number of spokes.

If two wheels have the same number of spokes, we can add distinct knots so the wheels can't be interchanged.

Any finite abelian group can be $\mathrm{TSG}(\Gamma)$

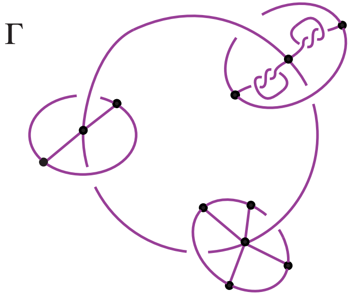
We can have any number of wheels with any number of spokes.

If two wheels have the same number of spokes, we can add distinct knots so the wheels can't be interchanged.



We can have any number of wheels with any number of spokes.

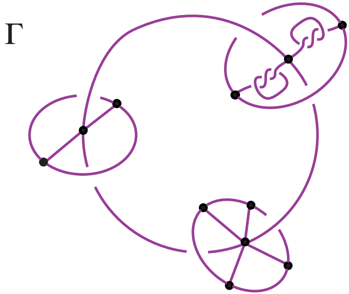
If two wheels have the same number of spokes, we can add distinct knots so the wheels can't be interchanged.



$\mathrm{TSG}(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$

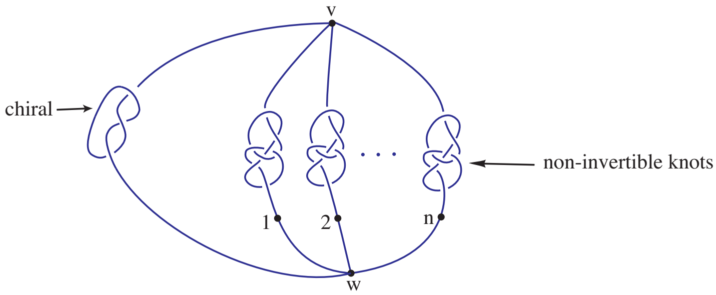
We can have any number of wheels with any number of spokes.

If two wheels have the same number of spokes, we can add distinct knots so the wheels can't be interchanged.



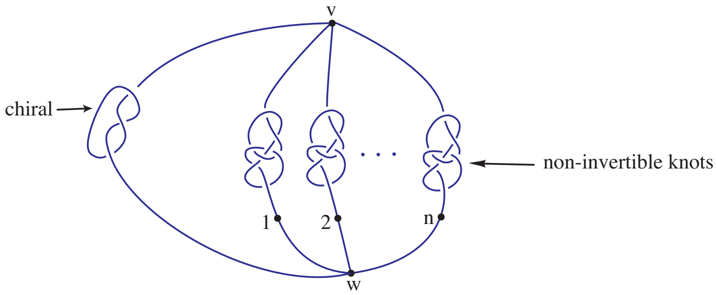
$\mathrm{TSG}(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$

Similarly, any finite abelian group can be $\mathrm{TSG}(\Gamma)$.



Non-invertible knots $\Longrightarrow$ v and w cannot be interchanged.

Chiral knot $\Longrightarrow$ there is no orientation reversing homeomorphism.



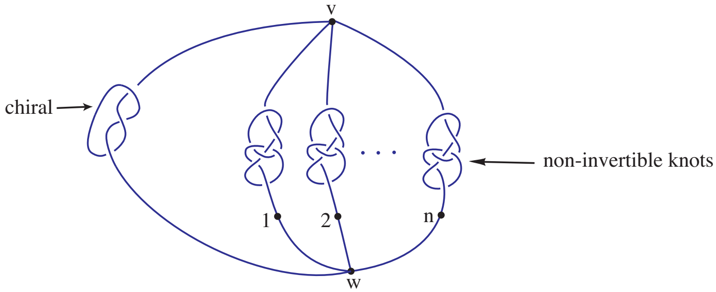
Non-invertible knots $\Longrightarrow$ v and w cannot be interchanged.

Chiral knot $\Longrightarrow$ there is no orientation reversing homeomorphism.

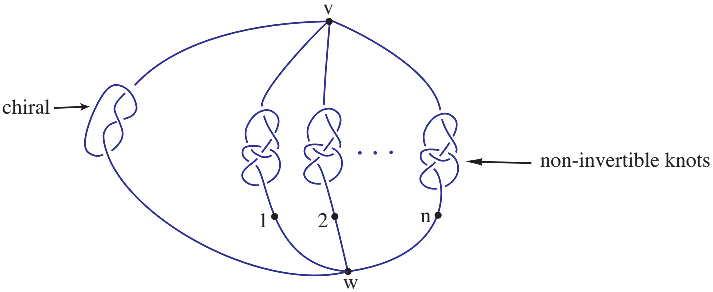
Any transposition (ij) is induced by twisting a pair of strands.

Thus $\mathrm{TSG}(\Gamma) = S_n$.

Can we get the alternating group A_n as $\mathrm{TSG}(\Gamma)$?



Can we get the alternating group A_n as $\mathrm{TSG}(\Gamma)$?



Theorem [Flapan, Naimi, Pommersheim, Tamvakis]

$\mathrm{TSG}(\Gamma)$ can be A_n iff $n \leq 5$.

Thus not every finite group can be $\mathrm{TSG}(\Gamma)$.

Definition

$\mathrm{TSG}_+(\Gamma)$ is the subgroup of $\mathrm{TSG}(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

Definition

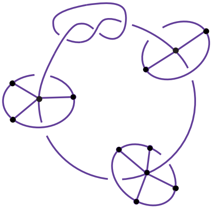
$\mathrm{TSG}_+(\Gamma)$ is the subgroup of $\mathrm{TSG}(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

$\mathrm{TSG}_+(\Gamma) = \mathrm{TSG}(\Gamma)$ or is a normal subgroup with index 2.

Definition

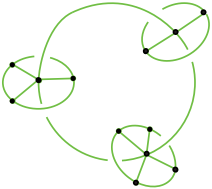
$\mathrm{TSG}_+(\Gamma)$ is the subgroup of $\mathrm{TSG}(\Gamma)$ induced by orientation preserving homeomorphisms of (S^3, Γ).

$\mathrm{TSG}_+(\Gamma) = \mathrm{TSG}(\Gamma)$ or is a normal subgroup with index 2.



$\mathrm{TSG}_+(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$

$\mathrm{TSG}(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$

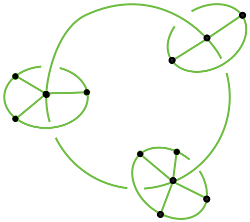


$\mathrm{TSG}_+(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$

$\mathrm{TSG}(\Gamma) = (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$

$\mathrm{TSG}_+(\Gamma)$ is not always induced by a finite subgroup of $\mathrm{Diff}_+(S^3)$ (group of orientation preserving diffeomorphisms of S^3).

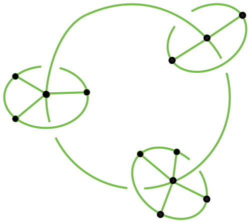
Example:



Spinning a wheel does not have finite order in S^3.

$\mathrm{TSG}_+(\Gamma)$ is not always induced by a finite subgroup of $\mathrm{Diff}_+(S^3)$ (group of orientation preserving diffeomorphisms of S^3).

Example:

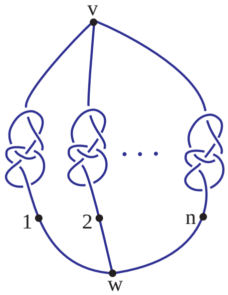


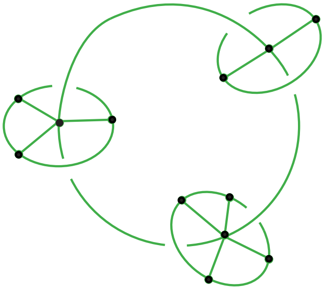
Spinning a wheel does not have finite order in S^3.

Not all finite abelian groups are subgroups of $\mathrm{Diff}_+(S^3)$. So $\mathrm{TSG}_+(\Gamma)$ need not even be isomorphic to a subgroup of $\mathrm{Diff}_+(S^3)$.

Definition

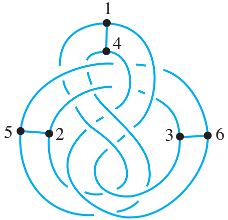
A graph γ is **3-connected** if at least 3 vertices together with their edges must be removed in order to disconnect γ or reduce it to a single vertex.



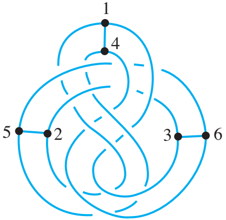


Neither of these graphs is 3-connected.

3-connected



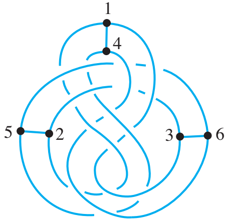
3-connected



(56)(23) is induced by turning the graph over.

(153426) is induced by slithering the graph along itself while interchanging the inner and outer knots.

3-connected

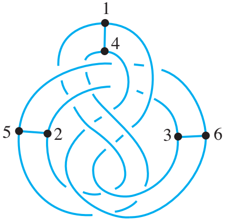


(56)(23) is induced by turning the graph over.

(153426) is induced by slithering the graph along itself while interchanging the inner and outer knots.

(153426) is not induced by a finite order homeomorphism of S^3, because a figure eight knot has no order 3 homeomorphism and cannot be the fixed point set of a finite order homeomorphism.

3-connected

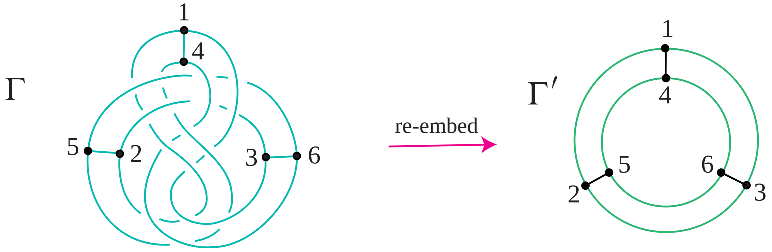


(56)(23) is induced by turning the graph over.

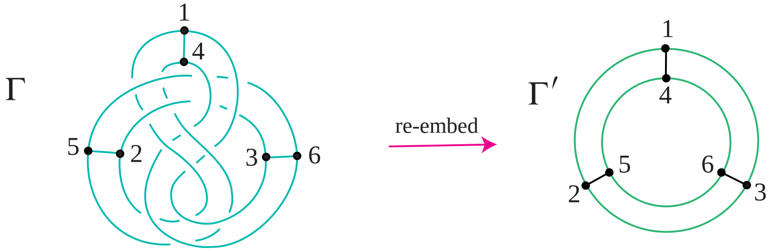
(153426) is induced by slithering the graph along itself while interchanging the inner and outer knots.

(153426) is not induced by a finite order homeomorphism of S^3, because a figure eight knot has no order 3 homeomorphism and cannot be the fixed point set of a finite order homeomorphism.

$\mathrm{TSG}_+(\Gamma) = \langle (56)(23), (153426) \rangle = D_6$. But is not induced by a finite group of homeomorphisms.



Γ' is a more symmetric embedding of Γ.



Γ′ is a more symmetric embedding of Γ.

(56)(23) is induced by turning Γ′ over left to right.

(153426) is induced by a glide rotation of Γ′ that interchanges the inner and outer circles while rotating counterclockwise.

$D_6 = \mathrm{TSG}_+(\Gamma')$ is induced by an isomorphic finite subgroup of $\mathrm{Diff}_+(S^3)$.

The theorem below shows that all 3-connected graphs have symmetric embeddings like the above example.

The theorem below shows that all 3-connected graphs have symmetric embeddings like the above example.

Theorem [Flapan, Naimi, Pommersheim, Tamvakis]

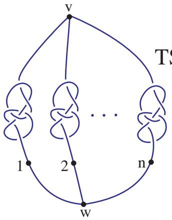
- Any 3-connected graph Γ embedded in S^3 can be re-embedded as Γ' so that $\mathrm{TSG}_+(\Gamma) \leq \mathrm{TSG}_+(\Gamma')$ and $\mathrm{TSG}_+(\Gamma')$ is induced by an isomorphic finite subgroup of $\mathrm{SO}(4)$.
- A finite group G is isomorphic to $\mathrm{TSG}_+(\Gamma)$ for some 3-connected embedded graph Γ iff G is isomorphic to a finite subgroup of $\mathrm{SO}(4)$.

$\mathrm{SO}(4) =$ the group of orientation preserving isometries of S^3.

Are subgroups of a given $\mathrm{TSG}_+(\Gamma)$ realizable?

Are subgroups of a given $\mathrm{TSG}_+(\Gamma)$ realizable?

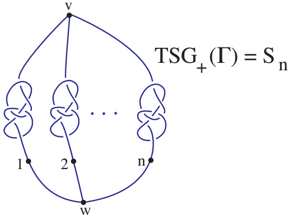
Recall:



$\mathrm{TSG}_+(\Gamma) = S_n$

But there is no embedding Γ' with $\mathrm{TSG}_+(\Gamma') = A_n$

Recall:

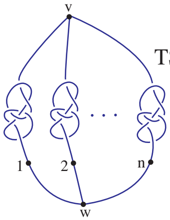


But there is no embedding Γ' with $\mathrm{TSG}_+(\Gamma') = A_n$

What if Γ is 3-connected?

Are subgroups of a given $\mathrm{TSG}_+(\Gamma)$ realizable?

Recall:



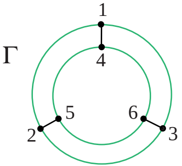
$\mathrm{TSG}_+(\Gamma) = S_n$

But there is no embedding Γ' with $\mathrm{TSG}_+(\Gamma') = A_n$

What if Γ is 3-connected?

Example:

$\mathrm{TSG}_+(\Gamma) = \langle (153426), (12)(45) \rangle = D_6$



Is every subgroup $H \leq D_6$ realizable by some embedding Γ'?
To answer this we use local knotting.

Splitting Spheres

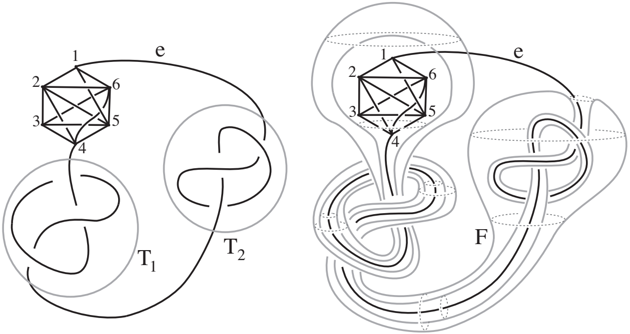
Schubert proved that every knot can be uniquely factored into prime knots.

Suzuki proved that every embedded graph can be split along spheres to obtain a unique collection of prime embedded graphs.

Schubert proved that every knot can be uniquely factored into prime knots.

Suzuki proved that every embedded graph can be split along spheres to obtain a unique collection of prime embedded graphs.

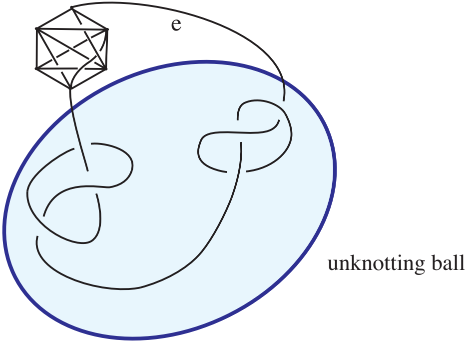
However, the splitting spheres are not necessarily unique up to an isotopy fixing the knot or graph.



F is not isotopic to T_1 or T_2

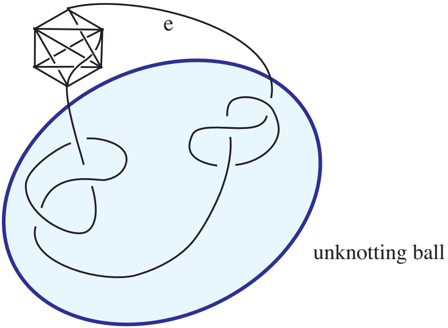
Definition

Let Γ be a 3-connected embedded graph. A ball meeting Γ in an arc containing all of the local knots of an edge e is an *unknotting ball* for e.



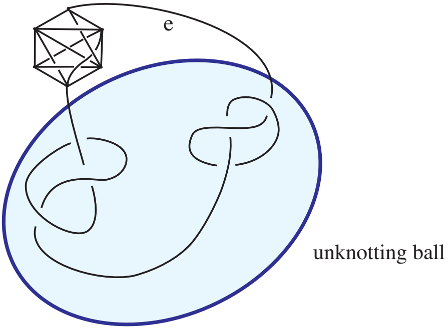
Uniqueness of Unknotting Balls [Flapan, Mellor, Naimi]

Let Γ be a 3-connected embedded graph. Then every locally knotted edge has an unknotting ball, and this ball is unique up to an isotopy of (S^3, Γ) fixing every vertex of Γ.



Uniqueness of Unknotting Balls [Flapan, Mellor, Naimi]

Let Γ be a 3-connected embedded graph. Then every locally knotted edge has an unknotting ball, and this ball is unique up to an isotopy of (S^3, Γ) fixing every vertex of Γ.



Proof uses JSJ characteristic decomposition of $S^3 - N(\Gamma)$

We use Uniqueness of Unknotting Balls to prove

Knot Addition Lemma [Flapan, Mellor, Naimi]

Let e be an edge of a 3-connected embedded graph Γ and $H \leq \mathrm{TSG}_+(\Gamma)$. Let K be a prime knot not in Γ which is invertible iff e is inverted by an element of H. Then K can be added to the edges in $\langle e \rangle_H$ to obtain Γ' such that $H \leq \mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$ and e is inverted by an element of $\mathrm{TSG}_+(\Gamma')$ iff e is inverted by an element of H.

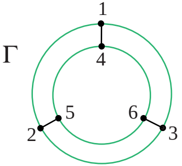
We use Uniqueness of Unknotting Balls to prove

Knot Addition Lemma [Flapan, Mellor, Naimi]

Let e be an edge of a 3-connected embedded graph Γ and $H \leq \mathrm{TSG}_+(\Gamma)$. Let K be a prime knot not in Γ which is invertible iff e is inverted by an element of H. Then K can be added to the edges in $\langle e \rangle_H$ to obtain Γ' such that $H \leq \mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$ and e is inverted by an element of $\mathrm{TSG}_+(\Gamma')$ iff e is inverted by an element of H.

Example:

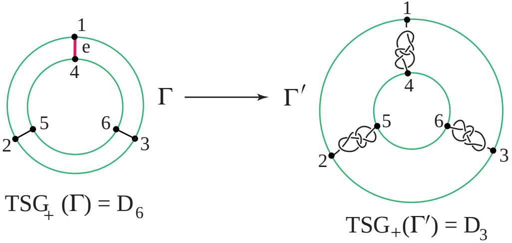
$\mathrm{TSG}_+(\Gamma) = <(153426), (12)(45)> = D_6$



Let $e = \overline{14}$ and $H = \langle (123) \rangle = \mathbb{Z}_3$. Then $\langle e \rangle_H = \{\overline{14}, \overline{25}, \overline{36}\}$.

$e = \overline{14}$ and $H = \langle(123)\rangle = \mathbb{Z}_3$. Then $\langle e\rangle_H = \{\overline{14}, \overline{25}, \overline{36}\}$.

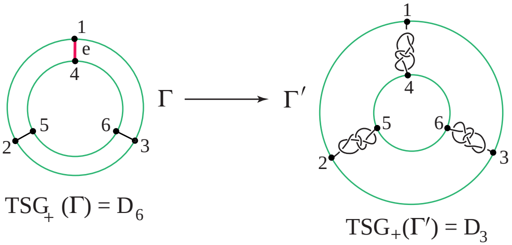
By Knot Addition Lemma:



Knot Addition Lemma $\implies H \leq \mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$

$e = \overline{14}$ and $H = \langle(123)\rangle = \mathbb{Z}_3$. Then $\langle e \rangle_H = \{\overline{14}, \overline{25}, \overline{36}\}$.

By Knot Addition Lemma:

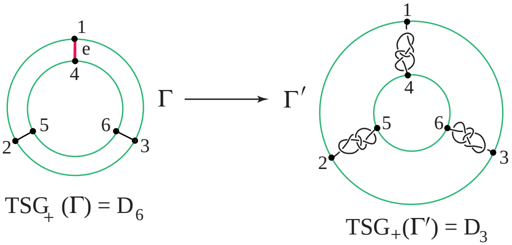


To prove $\mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$, we start with $g' : (S^3, \Gamma') \to (S^3, \Gamma')$ inducing some $\varphi \in \mathrm{TSG}_+(\Gamma')$.

Knot Addition Lemma $\implies H \leq \mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$

$e = \overline{14}$ and $H = \langle(123)\rangle = \mathbb{Z}_3$. Then $\langle e\rangle_H = \{\overline{14}, \overline{25}, \overline{36}\}$.

By Knot Addition Lemma:

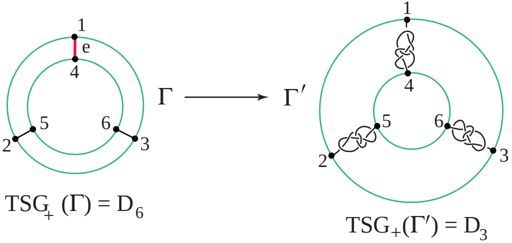


To prove $\mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$, we start with $g' : (S^3, \Gamma') \to (S^3, \Gamma')$ inducing some $\varphi \in \mathrm{TSG}_+(\Gamma')$.

By Uniqueness of Unknotting balls, we can assume g' leaves the set of unknotting balls for Γ' setwise invariant.

$e = \overline{14}$ and $H = \langle(123)\rangle = \mathbb{Z}_3$. Then $\langle e\rangle_H = \{\overline{14}, \overline{25}, \overline{36}\}$.

By Knot Addition Lemma:



To prove $\mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$, we start with $g' : (S^3, \Gamma') \to (S^3, \Gamma')$ inducing some $\varphi \in \mathrm{TSG}_+(\Gamma')$.

By Uniqueness of Unknotting balls, we can assume g' leaves the set of unknotting balls for Γ' setwise invariant.

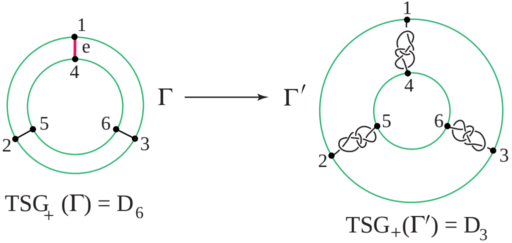
Then define $g : (S^3, \Gamma) \to (S^3, \Gamma)$ as g' outside of unknotting balls, and extend to Γ within these balls.

For a given $H \leq \mathrm{TSG}_+(\Gamma)$, can we re-embed Γ as Γ' such that $\mathrm{TSG}_+(\Gamma') = H$?

For a given $H \leq \mathrm{TSG}_+(\Gamma)$, can we re-embed Γ as Γ' such that $\mathrm{TSG}_+(\Gamma') = H$?

Example

$H = \langle (123) \rangle = \mathbb{Z}_3$ and $e = \overline{14}$. Then $\langle e \rangle_H = \{\overline{14}, \overline{25}, \overline{36}\}$.



By Knot Addition Lemma, we get

$$\mathbb{Z}_3 = H \leq \mathrm{TSG}_+(\Gamma') = D_3 \leq \mathrm{TSG}_+(\Gamma) = D_6$$

But we don't get $\mathrm{TSG}_+(\Gamma') = H$.

Sometimes every subgroup is realizable

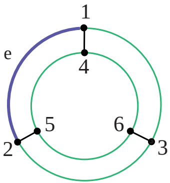
Subgroup Theorem [Flapan, Mellor, Naimi]

Let Γ be a 3-connected graph embedded in S^3 with an edge e that is not pointwise fixed by any non-trivial element of $\mathrm{TSG}_+(\Gamma)$. Then for every $H \leq \mathrm{TSG}_+(\Gamma)$, there is an embedding Γ' of Γ with $H = \mathrm{TSG}_+(\Gamma')$.

Subgroup Theorem [Flapan, Mellor, Naimi]

Let Γ be a 3-connected graph embedded in S^3 with an edge e that is not pointwise fixed by any non-trivial element of $\mathrm{TSG}_+(\Gamma)$. Then for every $H \leq \mathrm{TSG}_+(\Gamma)$, there is an embedding Γ' of Γ with $H = \mathrm{TSG}_+(\Gamma')$.

Example:



$\mathrm{TSG}_+(\Gamma) = D_6$

e is not pointwise fixed by any non-trivial element of $\mathrm{TSG}_+(\Gamma)$. Thus by Subgroup Theorem, for every $H \leq D_6$, there is some embedding Γ' of Γ with $\mathrm{TSG}_+(\Gamma') = H$.

We use the Subgroup Theorem to study topological symmetry groups of complete graphs.

We use the Subgroup Theorem to study topological symmetry groups of complete graphs.

Complete Graph Theorem [Flapan, Naimi, Tamvakis (2006)]

A finite group H is isomorphic to $\mathrm{TSG}_+(\Gamma)$ for some embedding Γ of a complete graph in S^3 if and only if H is a finite cyclic group, a dihedral group, a subgroup of $D_m \times D_m$ for some odd m, or A_4, S_4, or A_5.

We use the Subgroup Theorem to study topological symmetry groups of complete graphs.

Complete Graph Theorem [Flapan, Naimi, Tamvakis (2006)]

A finite group H is isomorphic to $\mathrm{TSG}_+(\Gamma)$ for some embedding Γ of a complete graph in S^3 if and only if H is a finite cyclic group, a dihedral group, a subgroup of $D_m \times D_m$ for some odd m, or A_4, S_4, or A_5.

However, for a given n, this theorem does not tell us the groups that can occur as $\mathrm{TSG}_+(\Gamma)$ for some embedding Γ of K_n.

We use the Subgroup Theorem to prove:

Theorem [Flapan, Mellor, Naimi]

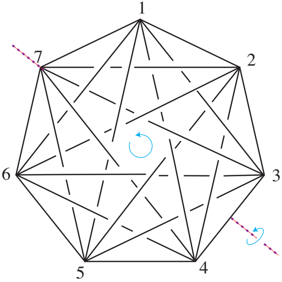
Let $n > 6$ and let Γ be an embedding of K_n in S^3 such that $\mathrm{TSG}_+(\Gamma)$ is a finite cyclic group, a dihedral group, or a subgroup of $D_m \times D_m$ for some odd m. Then for every $H \leq \mathrm{TSG}_+(\Gamma)$, there is an embedding Γ' of K_n such that $H = \mathrm{TSG}_+(\Gamma')$.

We use the Subgroup Theorem to prove:

Theorem [Flapan, Mellor, Naimi]

Let $n > 6$ and let Γ be an embedding of K_n in S^3 such that $\mathrm{TSG}_+(\Gamma)$ is a finite cyclic group, a dihedral group, or a subgroup of $D_m \times D_m$ for some odd m. Then for every $H \leq \mathrm{TSG}_+(\Gamma)$, there is an embedding Γ' of K_n such that $H = \mathrm{TSG}_+(\Gamma')$.

Example



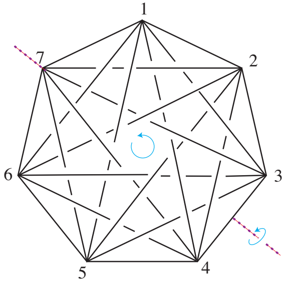
$\mathrm{TSG}_+(\Gamma) = D_7$

We use the Subgroup Theorem to prove:

Theorem [Flapan, Mellor, Naimi]

Let $n > 6$ and let Γ be an embedding of K_n in S^3 such that $\mathrm{TSG}_+(\Gamma)$ is a finite cyclic group, a dihedral group, or a subgroup of $D_m \times D_m$ for some odd m. Then for every $H \leq \mathrm{TSG}_+(\Gamma)$, there is an embedding Γ' of K_n such that $H = \mathrm{TSG}_+(\Gamma')$.

Example



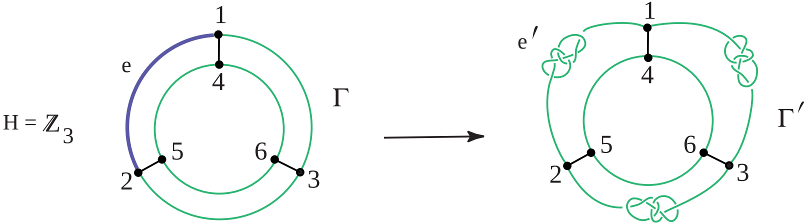
$\mathrm{TSG}_+(\Gamma) = D_7$

By above theorem, there is an embedding Γ' with $\mathrm{TSG}_+(\Gamma') = \mathbb{Z}_7$.

Suppose e is not fixed by any non-trivial element of $\mathrm{TSG}_+(\Gamma)$, $H \leq \mathrm{TSG}_+(\Gamma)$, and K a prime knot not contained in Γ which is invertible iff e is inverted by some element of H.

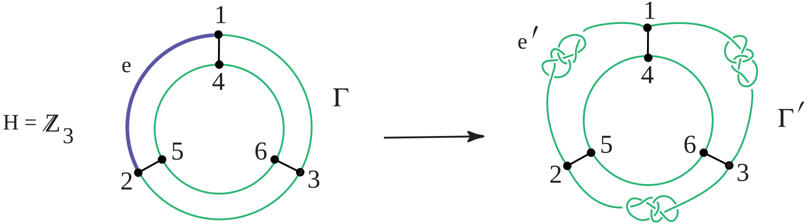
Suppose e is not fixed by any non-trivial element of $\mathrm{TSG}_+(\Gamma)$, $H \leq \mathrm{TSG}_+(\Gamma)$, and K a prime knot not contained in Γ which is invertible iff e is inverted by some element of H.

Use Knot Addition Lemma to add K to the edges in $\langle e \rangle_H$.



Suppose e is not fixed by any non-trivial element of $\mathrm{TSG}_+(\Gamma)$, $H \leq \mathrm{TSG}_+(\Gamma)$, and K a prime knot not contained in Γ which is invertible iff e is inverted by some element of H.

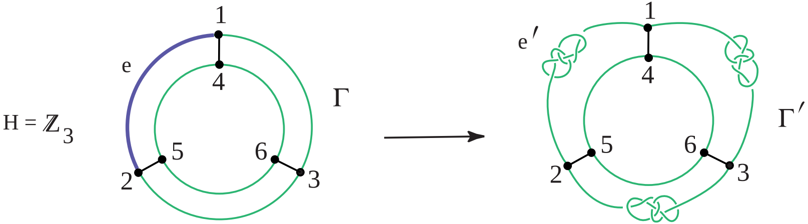
Use Knot Addition Lemma to add K to the edges in $\langle e \rangle_H$.



Then $H \leq \mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$ and e' is inverted by an element of $\mathrm{TSG}_+(\Gamma')$ iff e is inverted by an element of H.

Suppose e is not fixed by any non-trivial element of $\mathrm{TSG}_+(\Gamma)$, $H \leq \mathrm{TSG}_+(\Gamma)$, and K a prime knot not contained in Γ which is invertible iff e is inverted by some element of H.

Use Knot Addition Lemma to add K to the edges in $\langle e \rangle_H$.



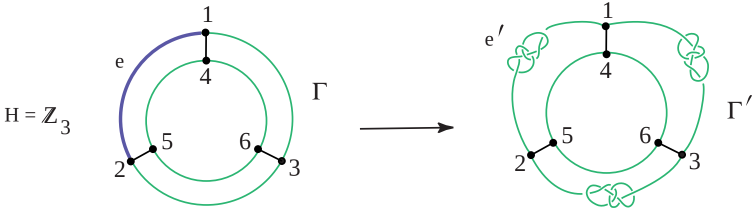
Then $H \leq \mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$ and e' is inverted by an element of $\mathrm{TSG}_+(\Gamma')$ iff e is inverted by an element of H.

To show $\mathrm{TSG}_+(\Gamma') \leq H$, let $\varphi \in \mathrm{TSG}_+(\Gamma')$.

$\varphi \in \mathrm{TSG}_+(\Gamma')$. Want to show $\varphi \in H$.

Recall:

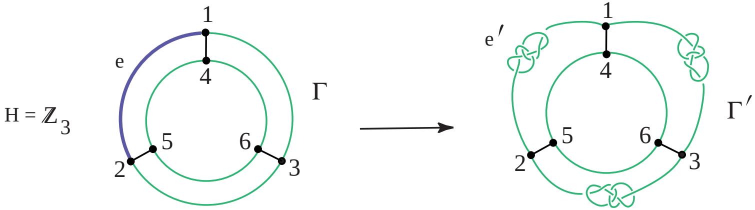
$H \leq \mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$ and e' is inverted by an element of $\mathrm{TSG}_+(\Gamma')$ iff e is inverted by an element of H.



$\varphi \in \mathrm{TSG}_+(\Gamma')$. Want to show $\varphi \in H$.

Recall:

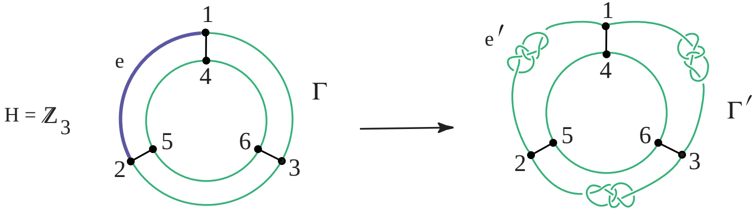
$H \leq \mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$ and e' is inverted by an element of $\mathrm{TSG}_+(\Gamma')$ iff e is inverted by an element of H.



e' contains K. Hence $\varphi(e')$ contains K.

Recall:

$H \leq \mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$ and e' is inverted by an element of $\mathrm{TSG}_+(\Gamma')$ iff e is inverted by an element of H.



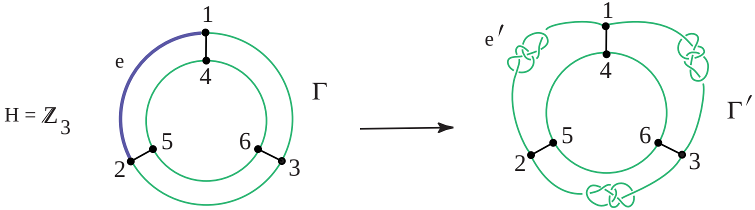
e' contains K. Hence $\varphi(e')$ contains K.

Only the edges in $\langle e' \rangle_H$ contain K. Hence $\varphi(e') \in \langle e' \rangle_H$.

$\varphi \in \mathrm{TSG}_+(\Gamma')$. Want to show $\varphi \in H$.

Recall:

$H \leq \mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$ and e' is inverted by an element of $\mathrm{TSG}_+(\Gamma')$ iff e is inverted by an element of H.



e' contains K. Hence $\varphi(e')$ contains K.

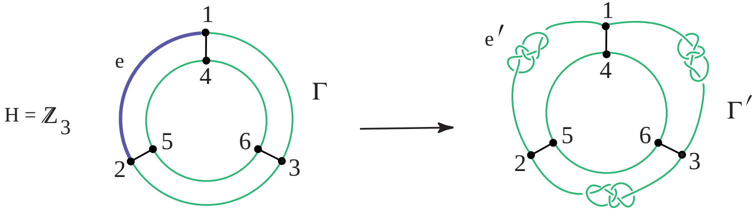
Only the edges in $\langle e' \rangle_H$ contain K. Hence $\varphi(e') \in \langle e' \rangle_H$.

So $\exists h \in H$ such that $h(e') = \varphi(e')$. Thus $h^{-1}\varphi$ setwise fixes e'.

$\varphi \in \mathrm{TSG}_+(\Gamma')$. Want to show $\varphi \in H$.

Recall:

$H \leq \mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$ and e' is inverted by an element of $\mathrm{TSG}_+(\Gamma')$ iff e is inverted by an element of H.



e' contains K. Hence $\varphi(e')$ contains K.

Only the edges in $\langle e' \rangle_H$ contain K. Hence $\varphi(e') \in \langle e' \rangle_H$.

So $\exists h \in H$ such that $h(e') = \varphi(e')$. Thus $h^{-1}\varphi$ setwise fixes e'.

Since $H \leq \mathrm{TSG}_+(\Gamma')$, $h^{-1}\varphi \in \mathrm{TSG}_+(\Gamma')$

$h^{-1}\varphi \in \mathrm{TSG}_+(\Gamma')$ setwise fixes e'. Want to show $\varphi \in H$.

Recall:

$H \leq \mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$ and e' is inverted by an element of $\mathrm{TSG}_+(\Gamma')$ iff e is inverted by an element of H.

$h^{-1}\varphi \in \mathrm{TSG}_+(\Gamma')$ setwise fixes e'. Want to show $\varphi \in H$.

Recall:

$H \leq \mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$ and e' is inverted by an element of $\mathrm{TSG}_+(\Gamma')$ iff e is inverted by an element of H.

If $h^{-1}\varphi$ doesn't invert e', then $g = h^{-1}\varphi \in \mathrm{TSG}_+(\Gamma)$ pointwise fixes e'.

If $h^{-1}\varphi$ inverts e', then e is inverted by some $f \in H$. So $g = fh^{-1}\varphi \in \mathrm{TSG}_+(\Gamma)$ pointwise fixes e'.

$h^{-1}\varphi \in \mathrm{TSG}_+(\Gamma')$ setwise fixes e'. Want to show $\varphi \in H$.

Recall:

$H \leq \mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$ and e' is inverted by an element of $\mathrm{TSG}_+(\Gamma')$ iff e is inverted by an element of H.

If $h^{-1}\varphi$ doesn't invert e', then $g = h^{-1}\varphi \in \mathrm{TSG}_+(\Gamma)$ pointwise fixes e'.

If $h^{-1}\varphi$ inverts e', then e is inverted by some $f \in H$. So $g = fh^{-1}\varphi \in \mathrm{TSG}_+(\Gamma)$ pointwise fixes e'.

In either case, $g =$ identity, since (by hypothesis) no non-trivial element of $\mathrm{TSG}_+(\Gamma)$ pointwise fixes e.

$h^{-1}\varphi \in \mathrm{TSG}_+(\Gamma')$ setwise fixes e'. Want to show $\varphi \in H$.

Recall:

$H \leq \mathrm{TSG}_+(\Gamma') \leq \mathrm{TSG}_+(\Gamma)$ and e' is inverted by an element of $\mathrm{TSG}_+(\Gamma')$ iff e is inverted by an element of H.

If $h^{-1}\varphi$ doesn't invert e', then $g = h^{-1}\varphi \in \mathrm{TSG}_+(\Gamma)$ pointwise fixes e'.

If $h^{-1}\varphi$ inverts e', then e is inverted by some $f \in H$. So $g = fh^{-1}\varphi \in \mathrm{TSG}_+(\Gamma)$ pointwise fixes e'.

In either case, $g =$ identity, since (by hypothesis) no non-trivial element of $\mathrm{TSG}_+(\Gamma)$ pointwise fixes e.

Thus either $\varphi = h \in H$ or $\varphi = f^{-1}h \in H$.

Hence $\varphi \in H$, and therefore $H = \mathrm{TSG}_+(\Gamma')$. ■