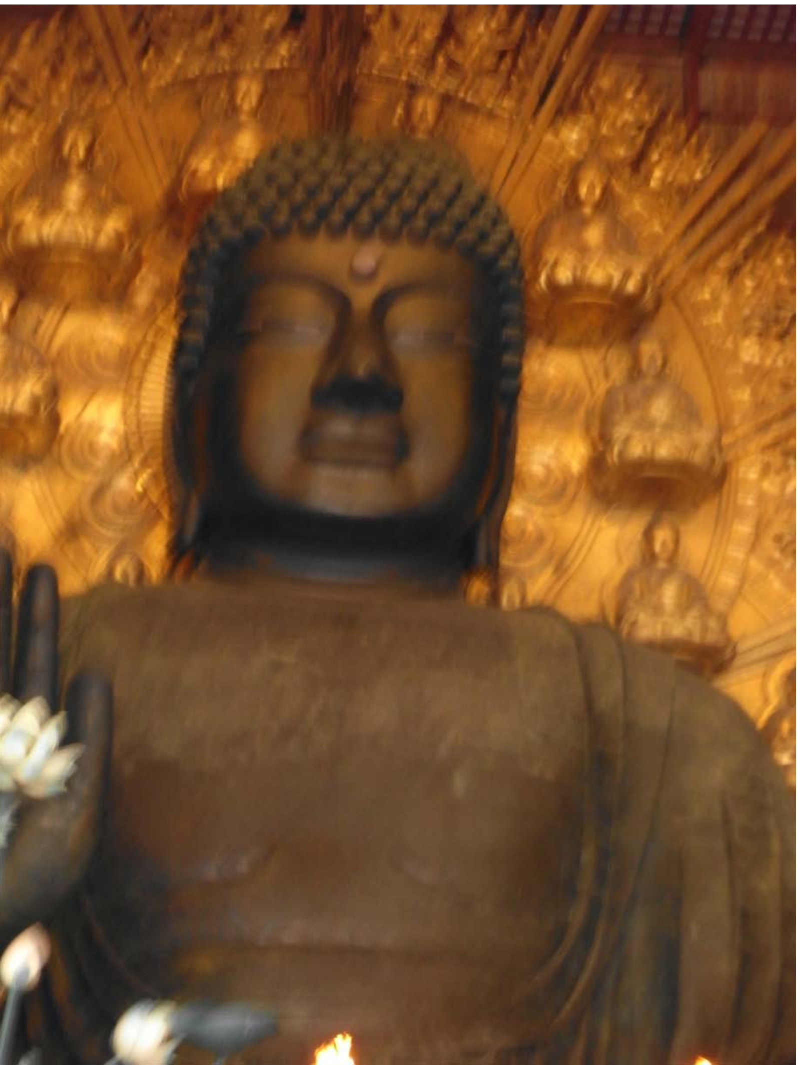
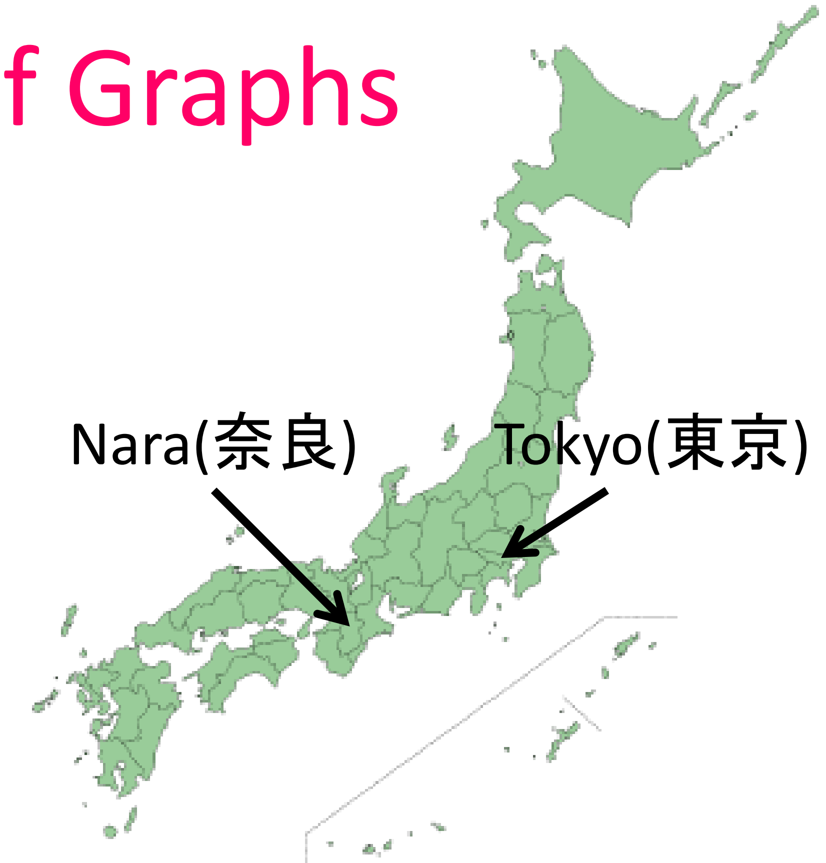
On Strongly Almost Trivial Embeddings of Graphs



The Great Buddha of Nara



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Contents

Definition & Notation

- G : finite graph
- f is a spatial embedding of G

  $\Leftrightarrow$ $f : G \to \mathbb{R}^3$: embedding

  - We call $f(G)$ a spatial graph
- f, f' : spatial embeddings of G

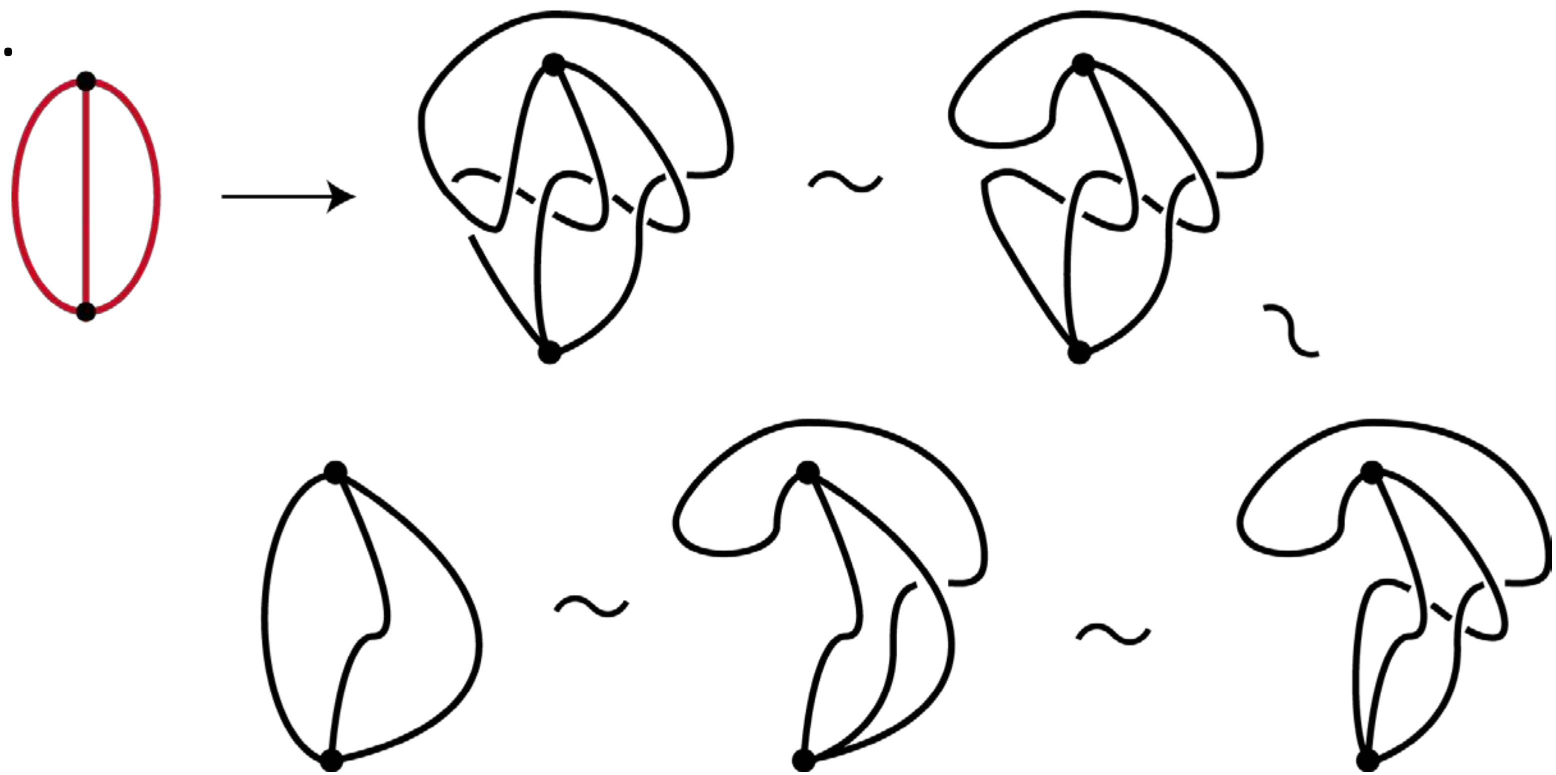
  f and f' are equivalent ($f \sim f'$)

  $\Leftrightarrow$ $\exists\, h : \mathbb{R}^3 \to \mathbb{R}^3$:(possibly orientation reversing) self-homeomorphism

  s.t. $h(f(G)) = f'(G)$

Definition & Notation

- f is trivial
  $\Leftrightarrow\ \exists f' \sim f$ s.t. $f'(G) \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$
  - Ex.



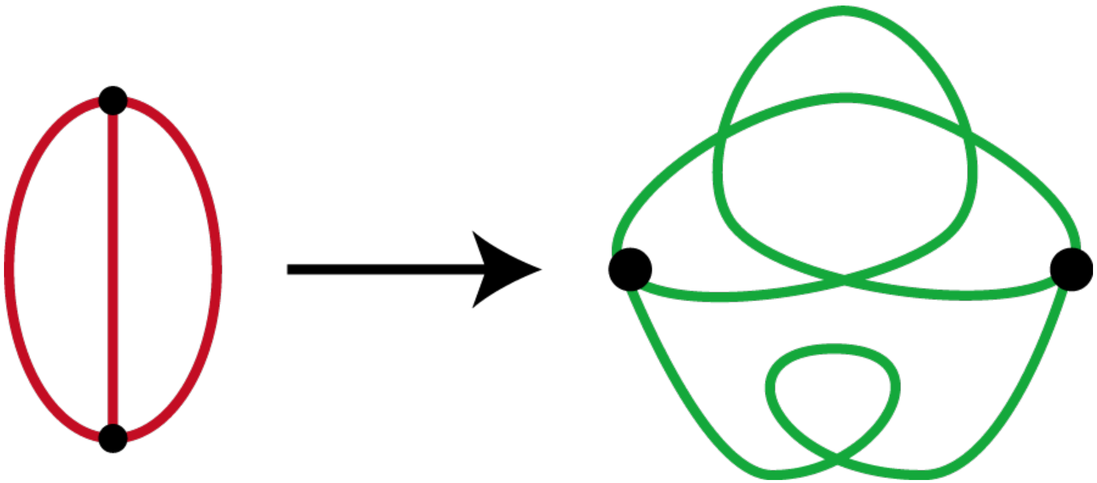
Definition & Notation

- G is planar
  $\Leftrightarrow\ \exists f : G \to R^2$: embedding

- Hence,
  G has a trivial embedding $\Leftrightarrow$ G is planar
- We consider only planar graphs.

Definition & Notation

- $\phi : G \to \mathbb{R}^2$: continuous map
- ϕ is a <u>projection</u> of G
  $\Leftrightarrow$ The multiple points of ϕ are only finitely many transversal double points away from vertices.
  - The image of a projection is also called a projection.



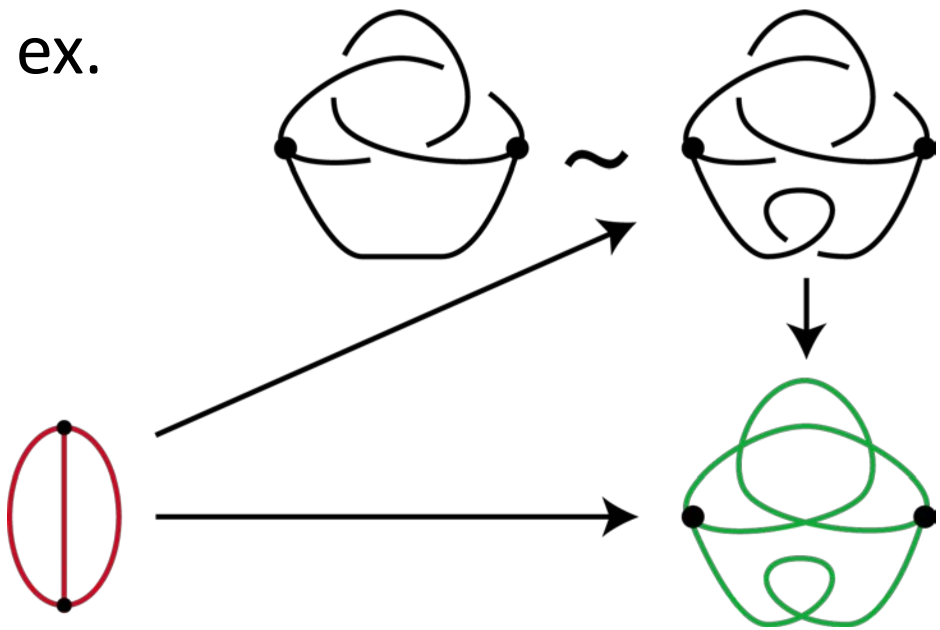
Definition & Notation

- ϕ is <u>a projection of a spatial embedding f</u>

  $\Leftrightarrow \ \exists f' \sim f$ s.t. $\phi = \pi \circ f'$

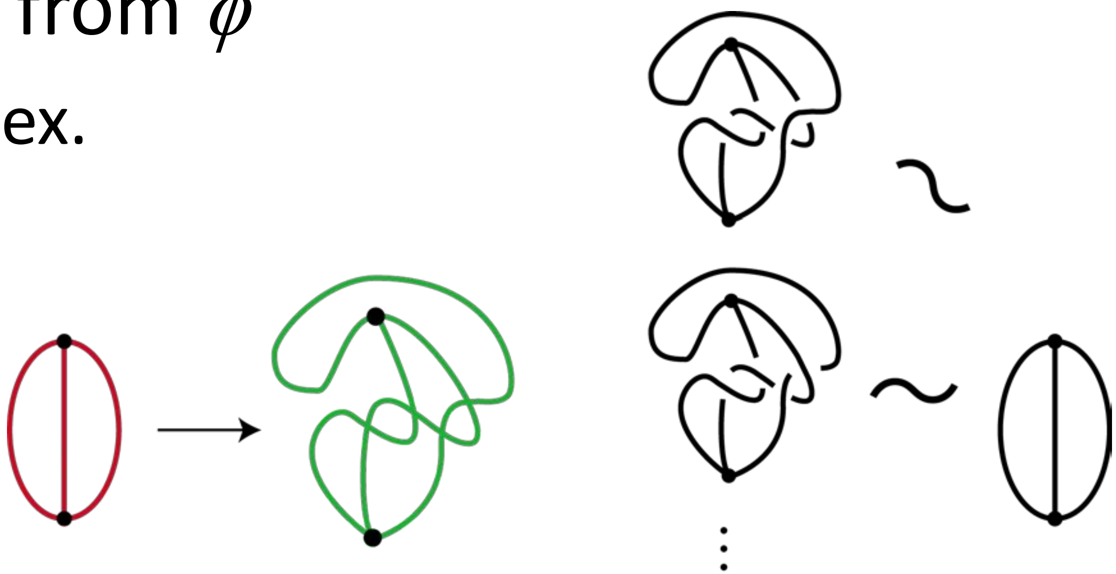
  - $\pi : R^3 \to R^2$ is a natural projection
  - We say f is obtained from ϕ
  - ex.

$$\begin{tikzcd} & R^3 \arrow[d, "\pi"] \\ G \arrow[ur, "f \sim f'"] \arrow[r, "\phi"'] & R^2 \end{tikzcd}$$



Definition & Notation

- projection ϕ is <u>trivial</u>
  $\Leftrightarrow$ only trivial spatial embeddings are obtained from ϕ
  - ex.



2^4 spatial embeddings are trivial

Definition & Notation

- G : planar graph
- f : a spatial embedding of G
- f is <u>almost trivial</u>
  $\Leftrightarrow$ $\forall H \subset G$ ($H \neq G$) : proper subgraph, $f|_H$ is trivial

- f is trivial $\Rightarrow$ f is almost trivial

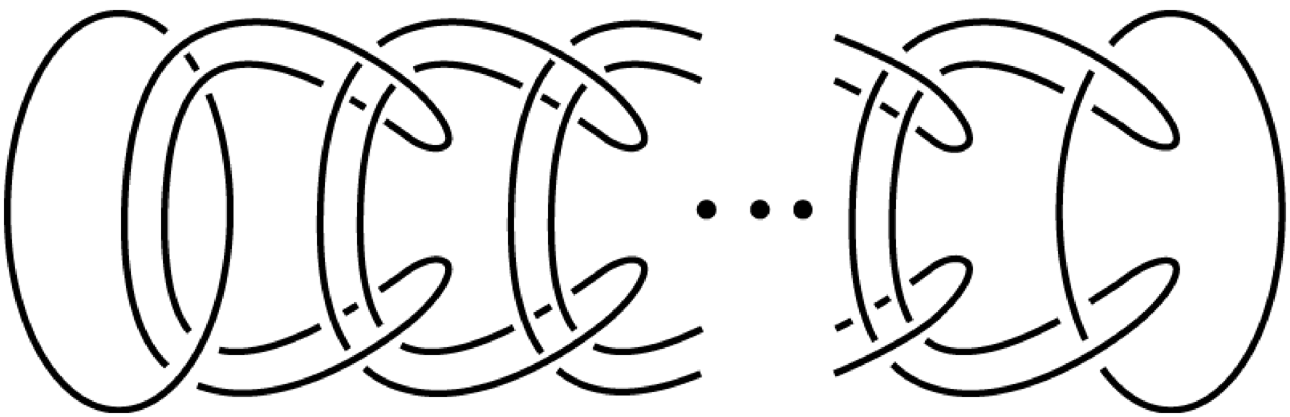
Definition & Notation

- G : planar graph
- f : a spatial embedding of G
- f is minimally knotted

  $\Leftrightarrow$ f is nontrivial

  $\forall H \subset G$ $(H \neq G)$: proper subgraph, $f|_H$ is trivial

  - ex. Brunnian link



Definition & Notation

- G : planar graph
- f : a spatial embedding of G
- f is <u>strongly almost trivial (SAT)</u>

  $\Leftrightarrow$ f is nontrivial

  $\exists\ \hat{f}$: projection of f s.t.

  $\forall H \subset G\ (H \neq G)$: proper subgraph, $\hat{f}|_H$ is trivial

  - We call $\hat{f}$ SAT projection.
- f is strongly almost trivial $\Rightarrow$ f is minimally knotted
- f is minimally knotted $\Rightarrow$ f is almost trivial

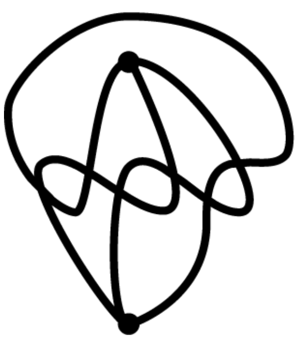
Definition & Notation

- ex. θ-curve has a SAT embedding

f

Kinoshita's θ-curve

$\hat{f}$



$\hat{f}|_H$

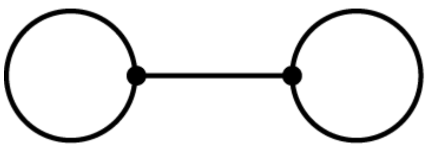
Known Results

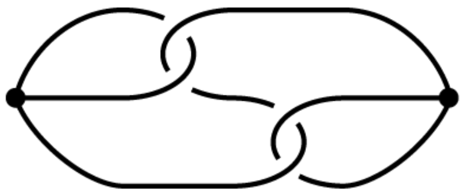
- $\forall G$: planar graph without vertices of degrees $\leqq 1$
  G has a minimally knotted spatial embedding
  - [Kawauchi, 1989], [Wu, 1993]

- $\exists G$: planar graph which has a SAT embedding
  $\exists G$: planar graph which does not have SAT embeddings

Known Results

- ex. handcuff graph has a SAT embedding





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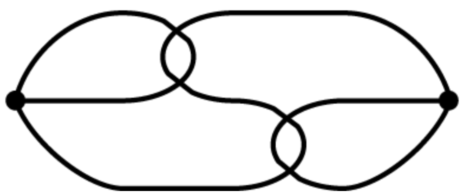
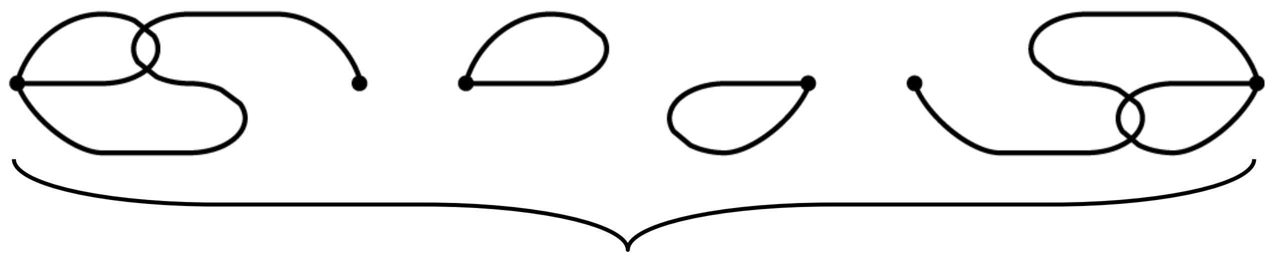


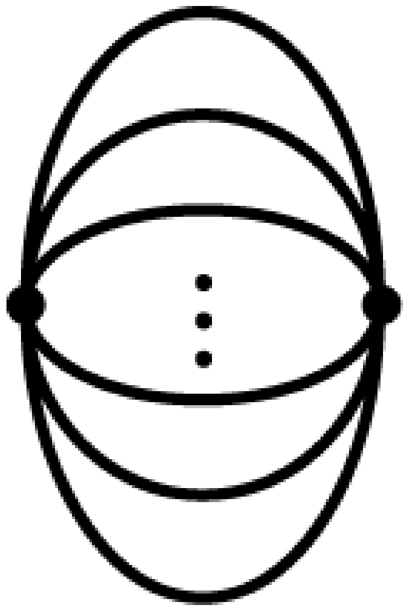
$\hat{f}$

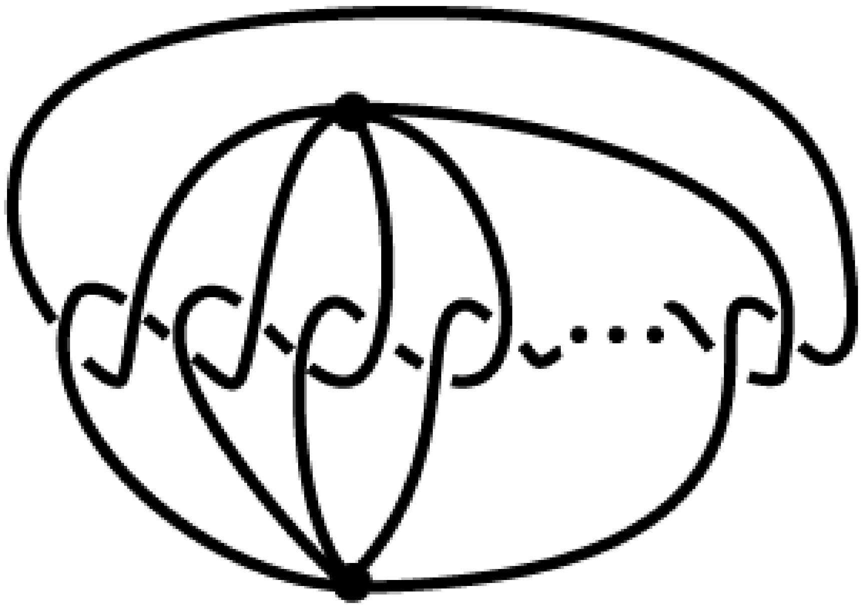


$\hat{f}|_H$

Known Results

- ex. θ_n-curve has a SAT embedding





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Known Results

- <u>Theorem 1</u> [Huh-Oh, 2003]

  G : connected planar graph without a cut vertex

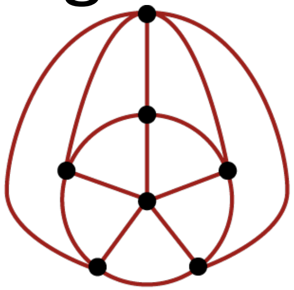
  G satisfies the following

  1. G has no multiple edges
  2. $\forall e_1, e_2 \in E(G)$ s.t. $e_1 \cap e_2 = \emptyset$,
     $\exists C_1, C_2$: disjoint cycles s.t. $e_1 \in E(C_1)$, $e_2 \in E(C_2)$
  3. $\forall e_1, e_2, e_3 \in E(G)$ s.t. $e_1 \cup e_2 \cup e_3$ is homeo. to a path
     $\exists C$: cycle s.t. $e_1, e_2, e_3 \in E(C)$

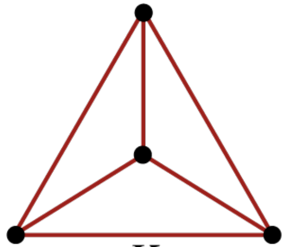
  $\Rightarrow$ G has no SAT embeddings

Known Results

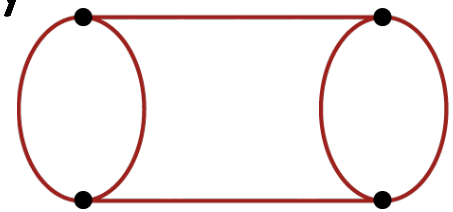
- ex. graphs which have no SAT embeddings
  - P_5 satisfies all assumptions of Thm 1.
  - K_4 does not satisfy the assumption 2 of Thm 1. [Huh-Oh, 2002]
  - Double-handcuff graph does not satisfy the assumptions 1 and 2 of Thm 1. [H, 2009]



P_5



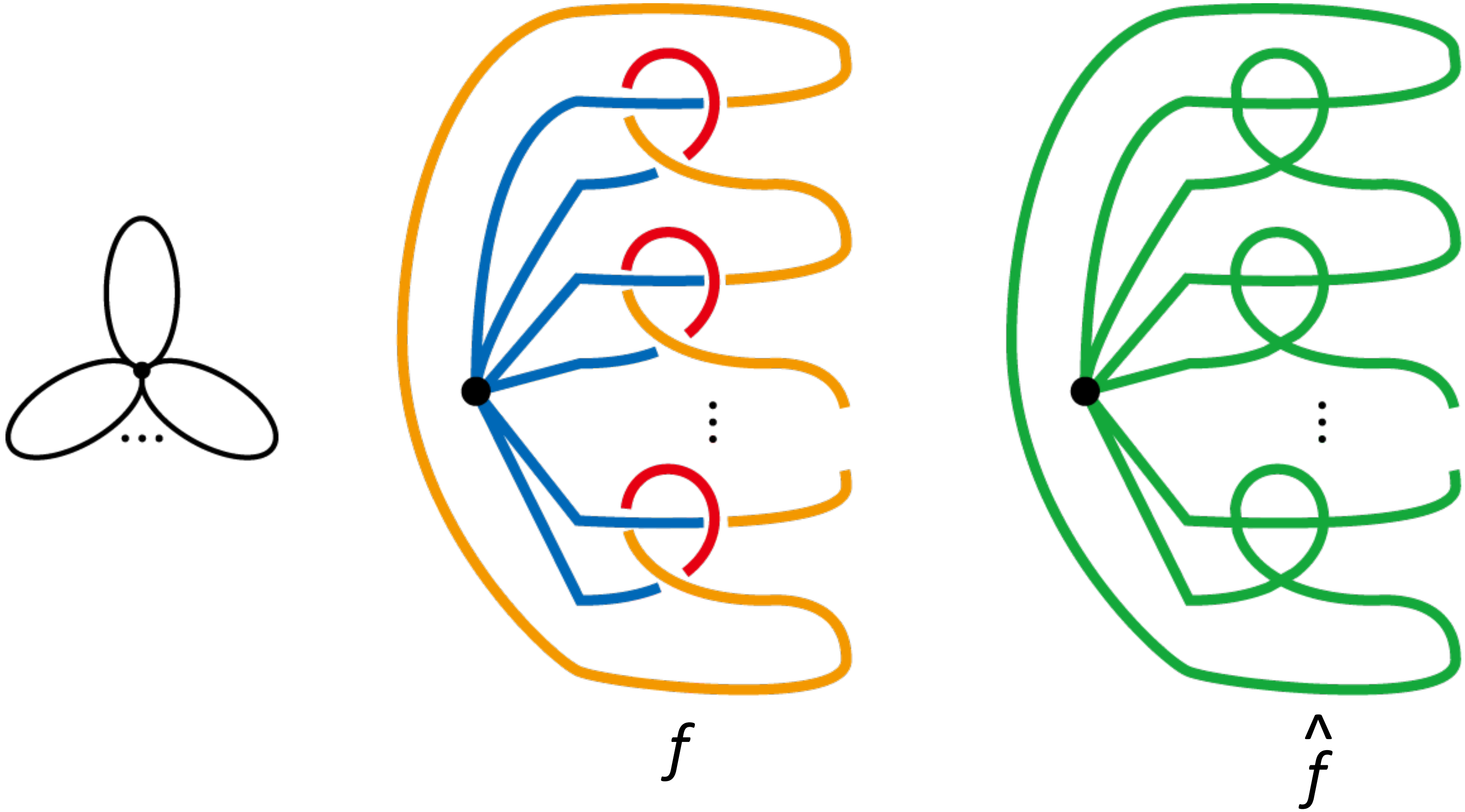
K_4



double-handcuff graph

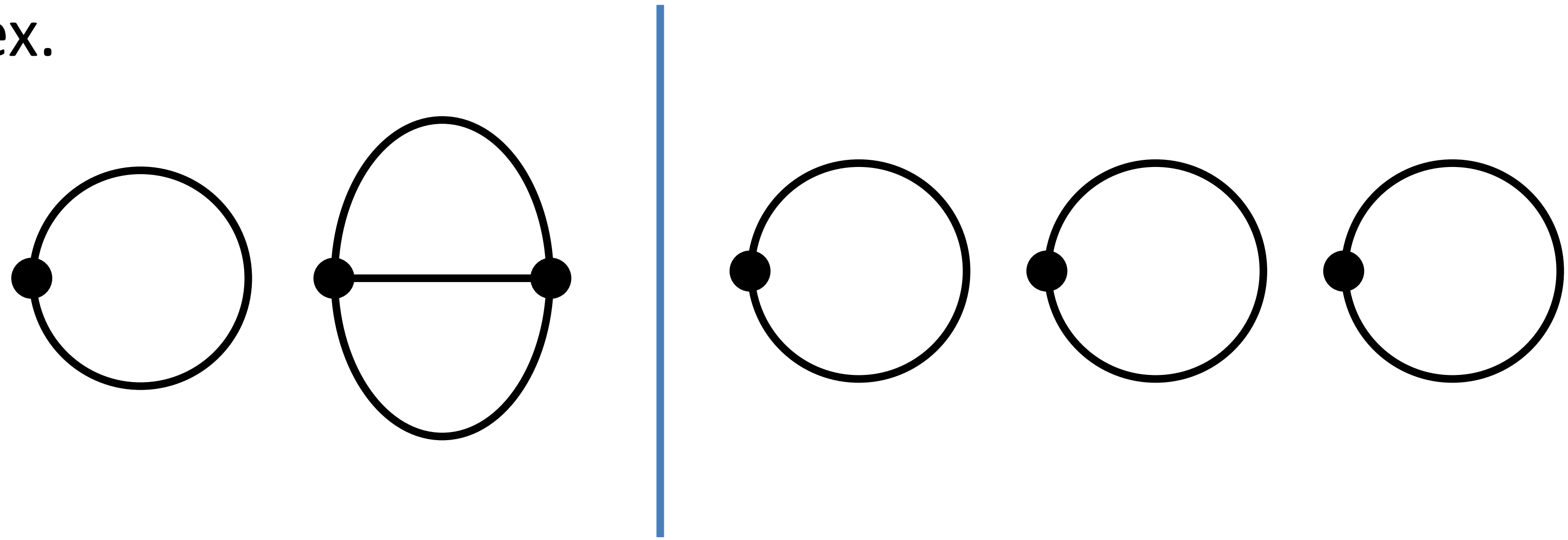
Main Results

- <u>Theorem 2</u> [H]
  n-bouquet has a SAT embedding



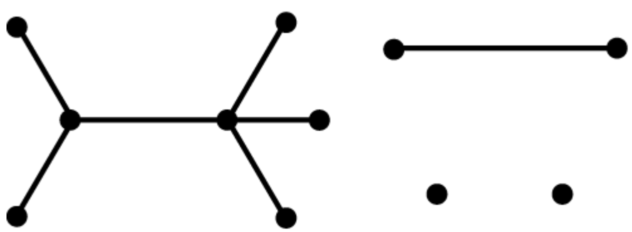
Main Results

- <u>Proposition 3</u> [H]
  G : disconneted graph without cut edges which
  is not homeo. to two disjoint circles
  $\Rightarrow$ G has no SAT embeddings
  - ex.

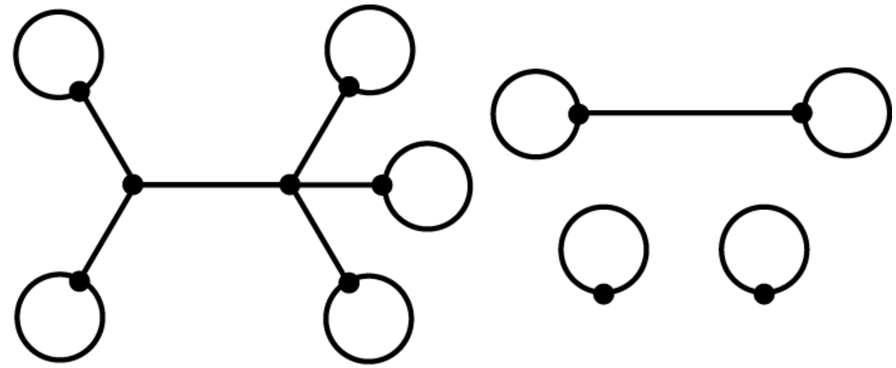


Main Results

- F : forest
- G_F : the graph obtained from F by adding a loop to the vertices v with $d_F(v) \leqq 1$
  - ex.



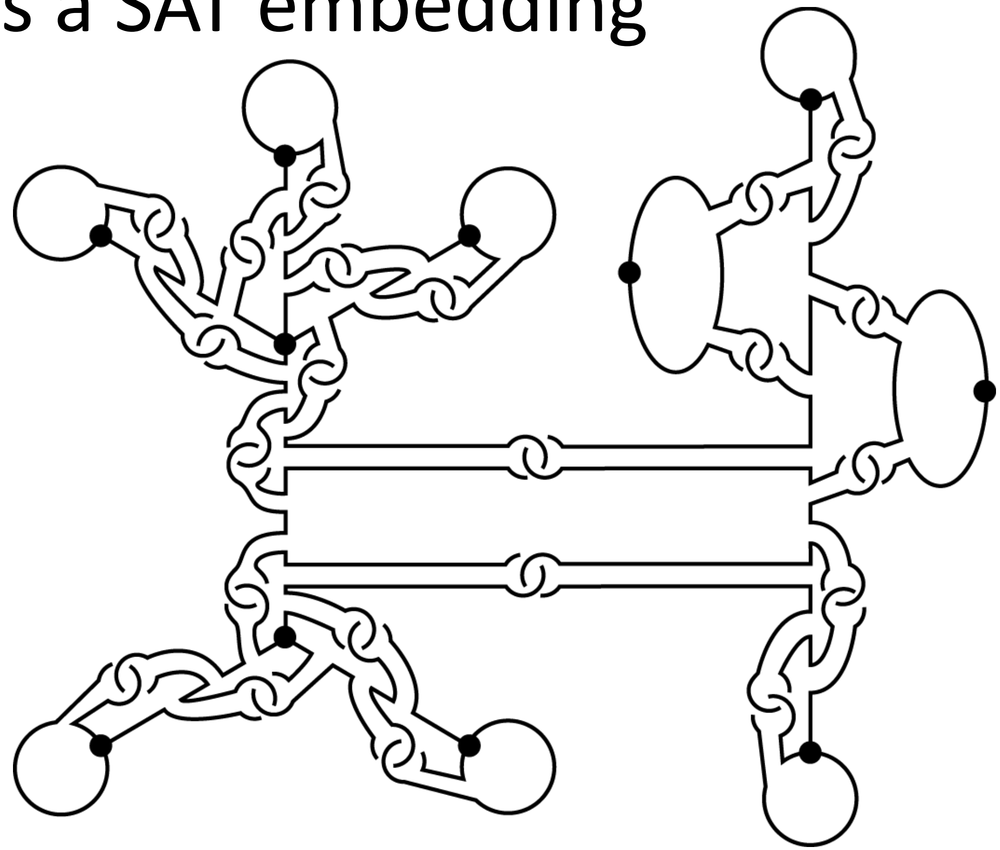
F



G_F

Main Results

- <u>Theorem 4</u> [H]
  F : forest with at least one edge
  $\Rightarrow$ G_F has a SAT embedding

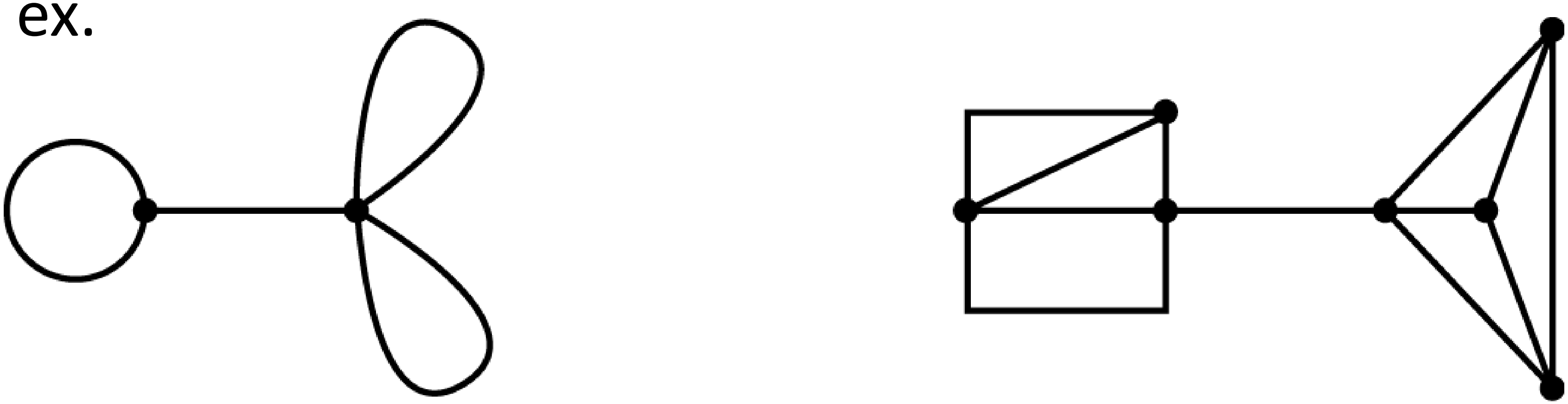


Main Results

- <u>Theorem 5</u> [H]

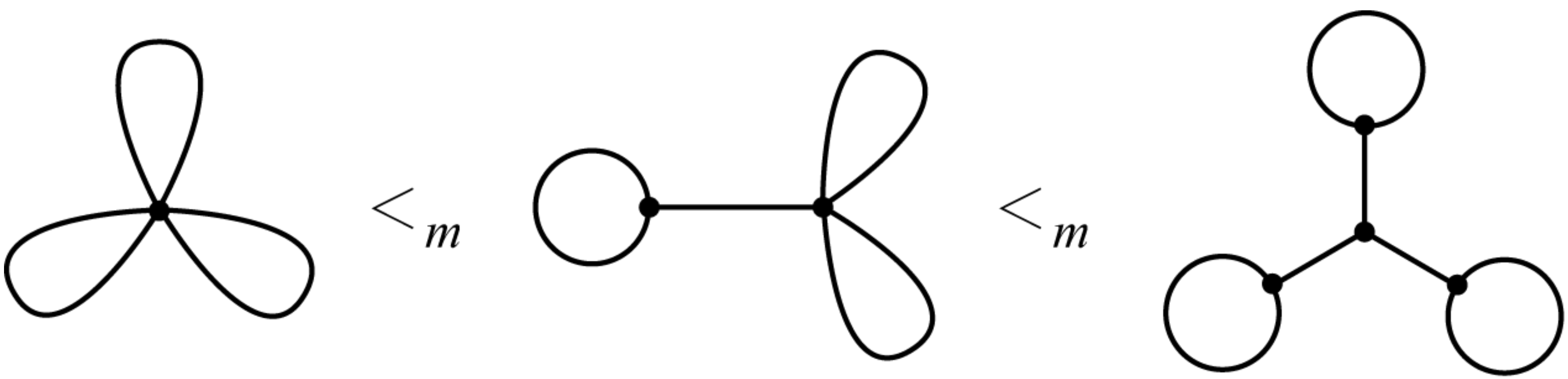
  G : connected graph with exactly one cut edge *e*
  s.t. *G* is not homeo. to a handcuff graph and
  each comp. of $G - e$ has at least one cycle
  ⇒ *G* has no SAT embeddings

  - ex.



Graph minors

- Corollary 6 [H]
  Both a property that a graph has a SAT embedding and a property that a graph has no SAT embeddings are not inherited by minors.



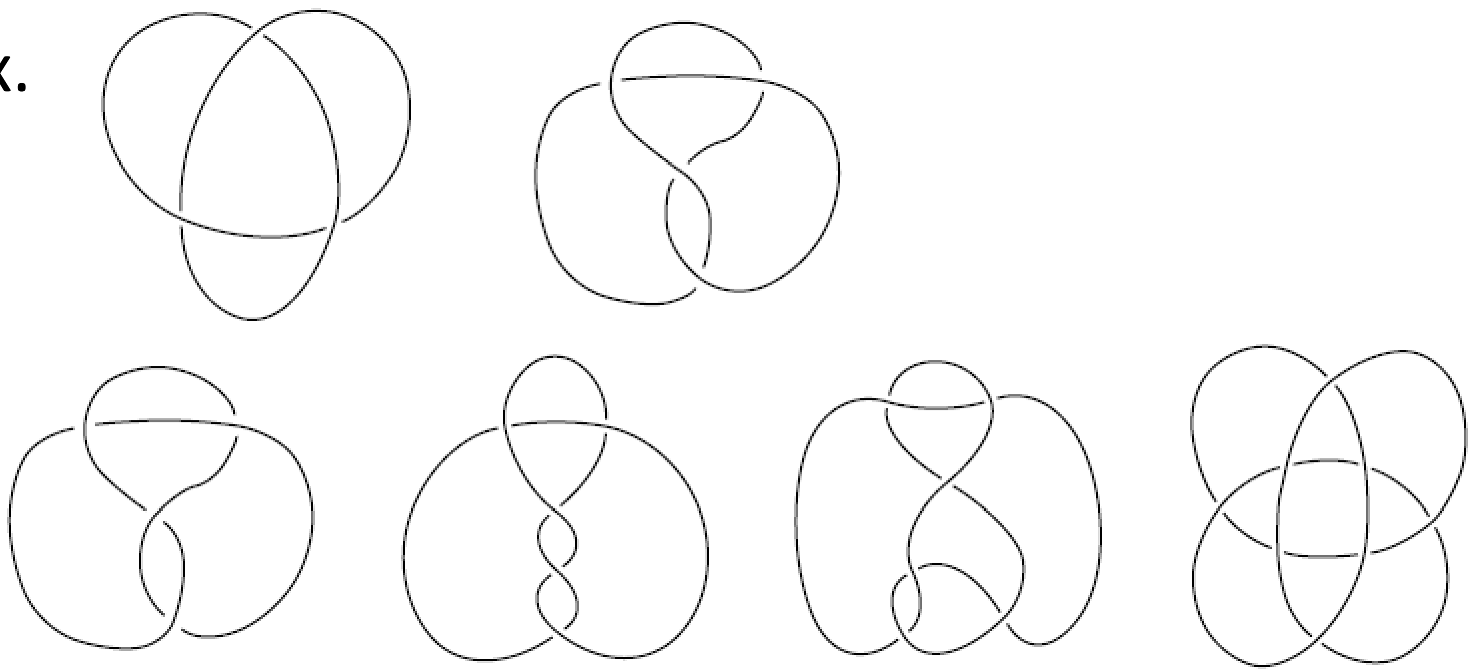
(a) SAT (b) no SAT (c) SAT

Related Topics

- A diagram D is <u>everywhere n-trivial</u>
  $\Leftrightarrow$ $\forall C$: subset of the set of crossings of D with n crossings
  the diagram obtained from D by switching over/under information at the crossings of C represents the trivial spatial graph
- A diagram D is <u>everywhere 1-trivial</u>
  $\Leftrightarrow$ Every diagram obtained from D by switching over/under information at one crossing of D represents the trivial spatial graph

Related Topics

- Askitas and Stoimenow conjecture that the only knots which have an everywhere 1-trivial diagram are the trivial knot, the trefoil knot and the figure eight-knot
  - ex.



Related Topics

- <u>Remark</u>
  These diagrams of SAT embeddings are everywhere 1-trivial.