Unraveling Tangles

Catherine Farkas

University of Illinois at Chicago

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Almost Unknotted Graphs

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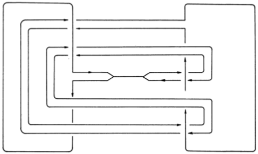

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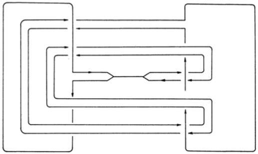

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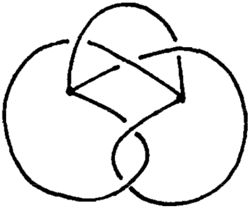

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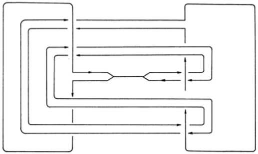

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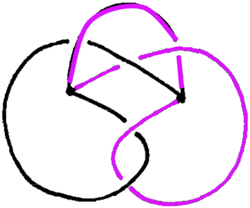

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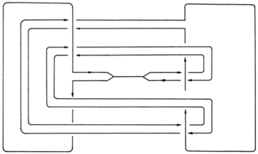

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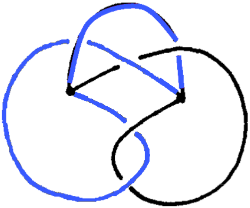

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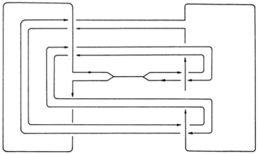

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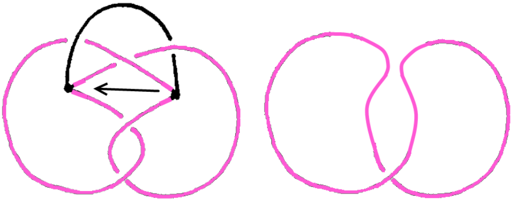
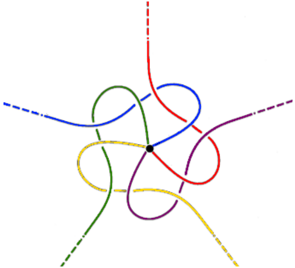


Almost Unknotted θ_n curves

- Suzuki [9] generalized Kinoshita's example to a family of almost unknotted θ_n graphs for every n .



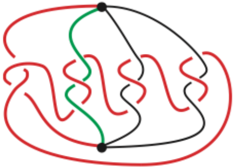
- Scharlemann [7] and Livingston [5] each reproved Suzuki's result using topological and geometric arguments; McAtee, Silver, and Williams [6] reproved the result more recently using graph colorings

Chirality of Almost Unknotted θ_n curves

- Walcott [11] showed that Kinoshita's θ_3 curve is chiral (not isotopic to its mirror image)
- Ushijima [10] showed that all of Suzuki's θ_n curves are chiral
- Hara [3] found two families of locally unknotted (no edge contains a knot) θ_4 curves where all the θ_4 curves in one family are chiral and all the curves in the other are achiral

Chirality of Almost Unknotted θ_n curves

- Hara's graphs are locally unknotted but not all are almost unknotted



Chirality of Almost Unknotted θ_n curves

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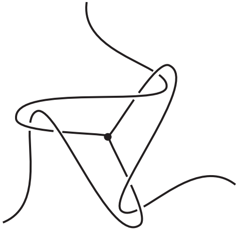
Hara's test for planarity

Let Γ be a θ_4 graph. Then Γ is planar if and only if replacing each vertex of Γ by any rational tangle results in a knot or link whose bridge index is no greater than 2.

What is a ravel?

- Molecular knots and links have been chemically synthesized: *proteins, organometallic compounds, DNA*
- Chemists are interested in such molecules because topological characteristics can have an effect on molecular and biological behavior
- Recently, chemists became interested in the structure of an "*n-ravel*", which they defined as "an entanglement of n edges around a vertex that contains no knots or links" [2]



A 3-ravel

What is a Ravel?

Mathematical Definition

An n-ravel is an embedded θ_n graph that:

(i) *contains no nontrivial knots*

(ii) *cannot be deformed to a planar graph*

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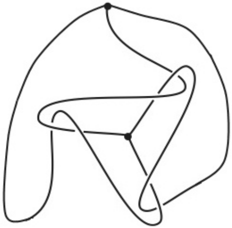
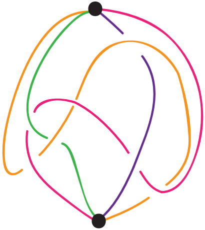


3-ravel



4-ravel

Question of interest

Recall:

Hara's test for planarity

Let Γ be a θ_4 graph. Then Γ is planar if and only if replacing each vertex of Γ by a rational tangle results in a knot or link whose bridge index is no greater than 2.

- By contrast, we begin with a projection of an algebraic tangle and replace a crossing with a vertex. We bring the vertices together in the boundary of the tangle ball. Can the resulting tangle be a 4-ravel?
- If a θ_4 graph is not a ravel then it is also not almost unknotted.

Rational Tangles

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A 2-string tangle is called rational if it can be deformed to a trivial tangle when the endpoints are free to move within the boundary of the tangle ball.

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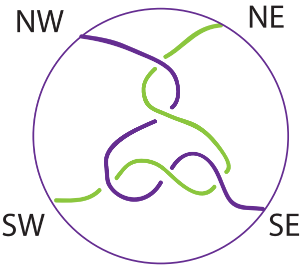
Example



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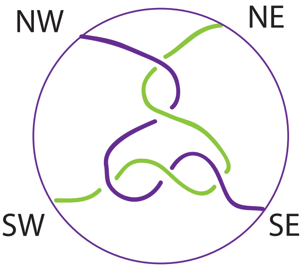
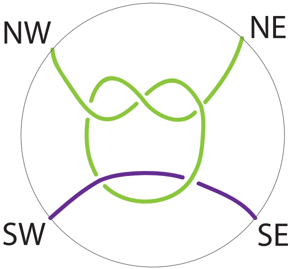
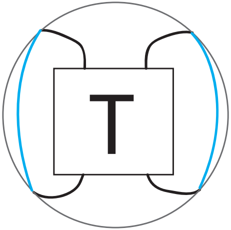


Nonexample

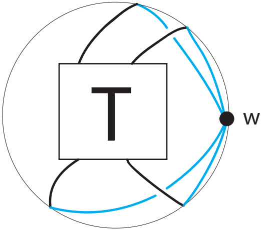


More definitions

The denominator closure and vertex closure of a tangle are defined as shown below, where the blue arcs are in the boundary of the ball:



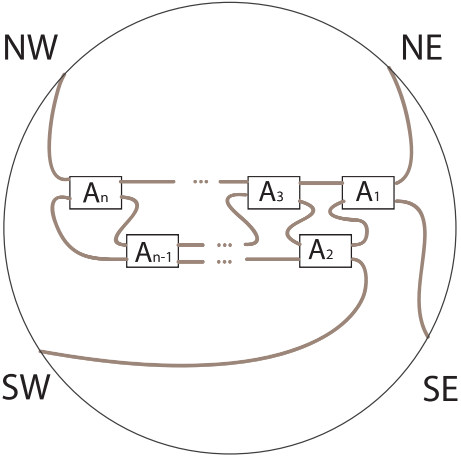
Denominator closure



Vertex closure

General form of rational tangle

- Alternating, 3-braid representation where each A_i represents a row of twists and the crossings alternate from one row to the next

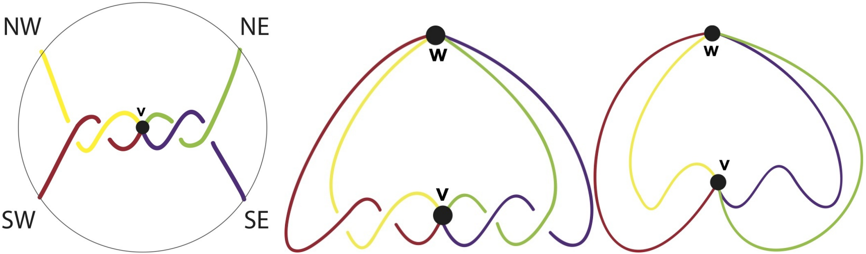


Statement of Theorem

Rational Tangle Theorem (Farkas, Flapan, Sullivan)

Let A be a projection of a rational 2-string tangle in alternating 3-braid representation, and let A' be obtained by replacing a crossing of A with a vertex. Then the vertex closure of A' is either planar or contains a knot, and therefore is not a 4-ravel.

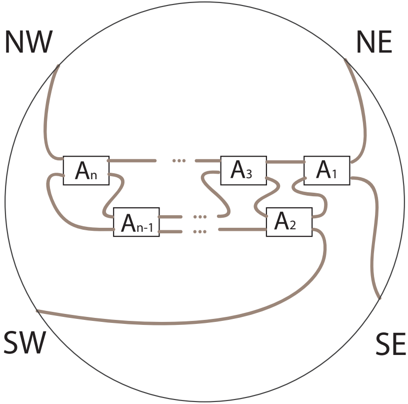
Example:



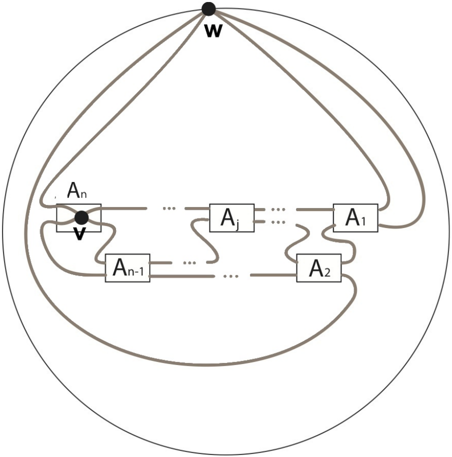
Outline of Proof of Theorem

3 Cases:

- the vertex replaces a crossing in A_n,
- the vertex replaces a crossing in A_{n-1},
- the vertex replaces a crossing anywhere else



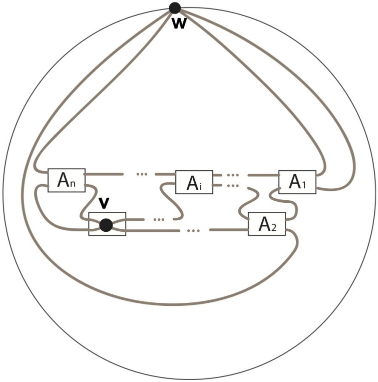
Case 1: The vertex replaces a crossing in A_n



- Untwist the crossings in A_1.
- Once the crossings in A_1 have been removed, we can untwist the crossings in A_2.
- Continue this process until all crossings have been removed. Thus, the graph is planar.

Case 2: The vertex replaces a crossing in A_{n-1}

- If A_n has only one crossing, then we can deform the graph to a plane using a similar technique to that used in Case 1.



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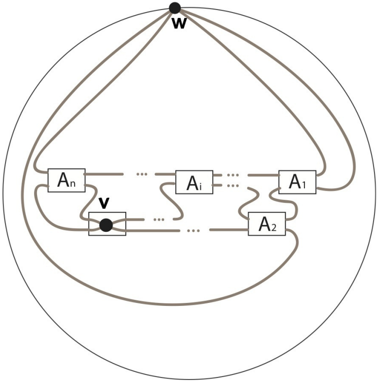
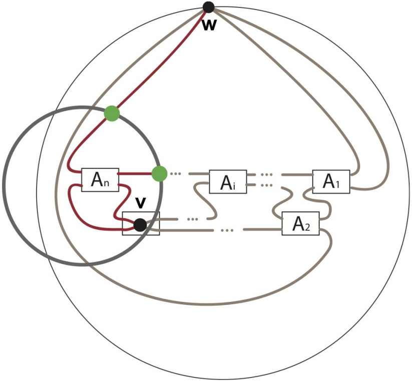


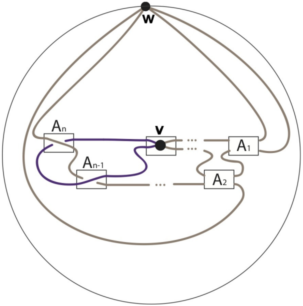
- We show that if A_n has more than one crossing, then either the graph is not a θ_4 graph or it contains a knot.



Case 3: The vertex is not in A_n or A_{n-1}

Consider the subtangle of A containing $A_n, A_{n-1}, \ldots, A_{i+1}$ where the vertex v is in A_i:

- If the subtangle only has two crossings, then the original graph is not a θ_4 graph, and therefore is not a ravel.



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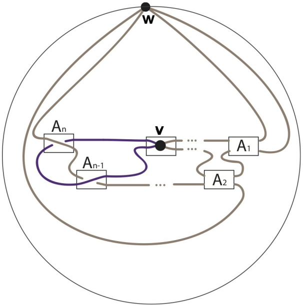
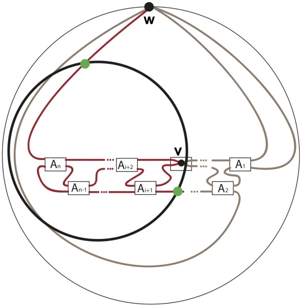
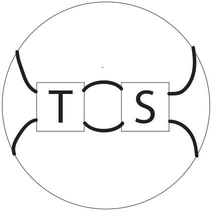


- If the subtangle has three or more crossings and is a θ_4 graph, then the graph contains a knot and is not a ravel.



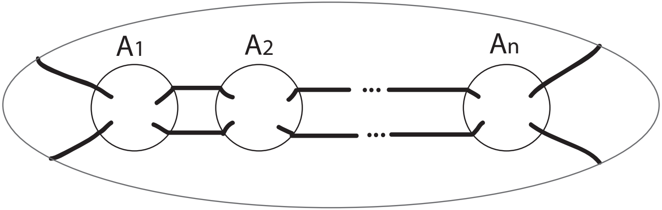
Definition

Tangles can be added: $T + S =$



Definition

A Montesinos tangle is the sum of a finite number of rational tangles.



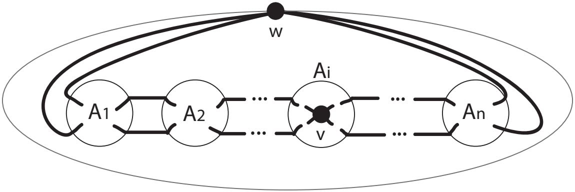
Definition

A projection of a Montesinos tangle A is said to be in reduced form if it is written as a sum of a minimal number of rational tangles (over all possible projections).

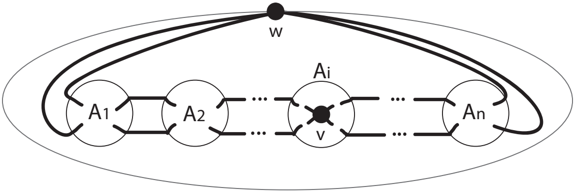
Theorem for Montesinos Tangles

Montesinos Tangle Theorem (Farkas, Flapan, Sullivan)

Given a projection A of a Montesinos tangle $A = A_1 + A_2 + \cdots + A_n$ in reduced form where each A_i is a rational tangle and A itself is not rational, let A' be obtained by replacing a crossing of A with a vertex. If the vertex closure of A' is a θ_4 graph, then it contains a knot and therefore is not a ravel.



Note: If the vertex replaces a crossing in A_i and the vertex closure of A' is a θ_4 graph, then the denominator closure of A_k is a (possibly trivial) knot for all $k \neq i$.

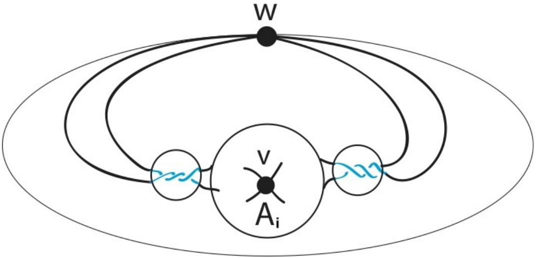


The vertex closure of A'

That is, the denominator closure of each A_k is forced to have only one component.

Outline of Proof

Suppose that for some k the denominator closure of A_k is a trivial knot.

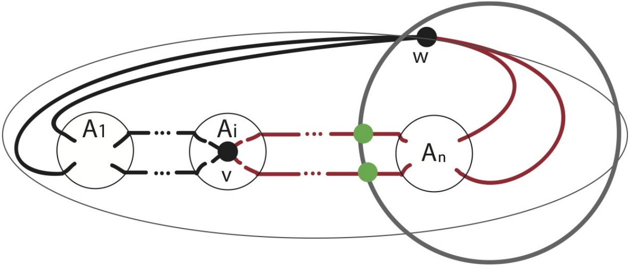


Then, A is not in reduced form, contrary to our hypothesis.

Outline of proof

Therefore, the denominator closure of every A_k is a nontrivial knot.

In this case, we can show the vertex closure of A' contains a knot and therefore is not a ravel.



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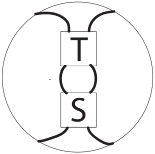
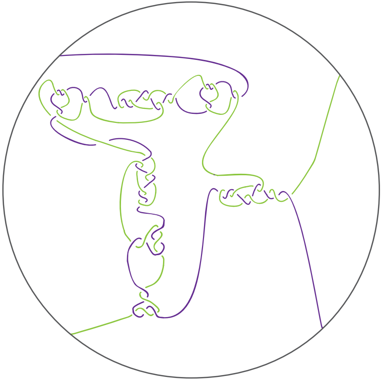


$T \times S$

Example:



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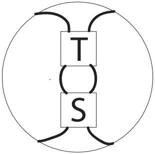

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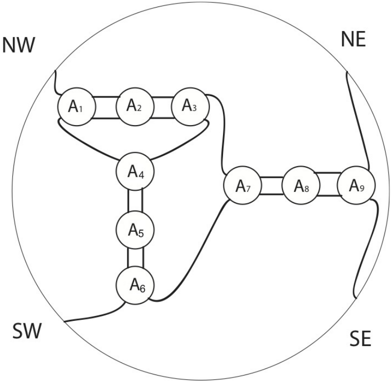

$$[(A_1 + A_2 + A_3) \times A_4 \times A_5 \times A_6] + A_7 + A_8 + A_9$$

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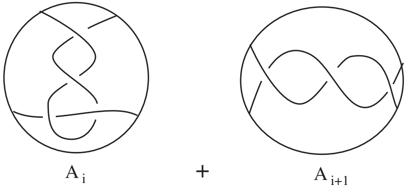
Ex. A tangle containing this $A_i + A_{i+1}$ is not in reduced form.



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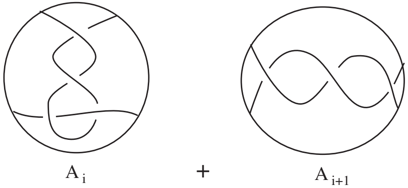
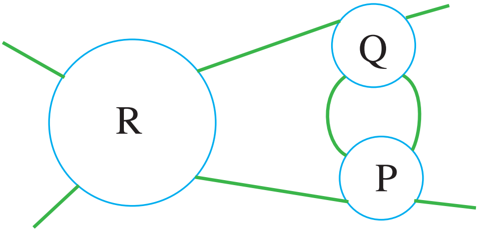


- $((A_1 + A_2) + (A_3 \times A_4)) \times A_5$ is not in reduced form since we can rewrite it as $(A_1 + A_2 + (A_3 \times A_4)) \times A_5$.

Definition

Let A be a projection of an algebraic tangle in reduced form containing:
$(R + (Q \times P))$, $((Q \times P) + R)$, $(R \times (Q + P))$, or $((Q + P) \times R)$, where R and at least one of P or Q is rational. Then we say A contains a *bad triangle*.

$$(R + (Q \times P))$$

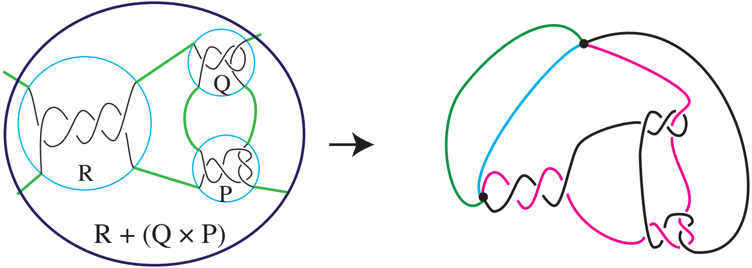


Theorem for Algebraic Tangles

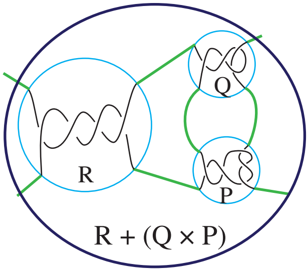
Theorem (Farkas, Flapan, Sullivan)

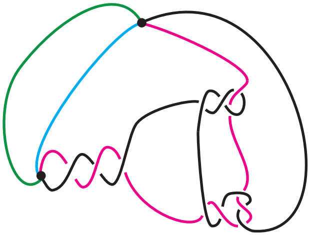
Let A be a projection of a nonrational algebraic tangle in reduced form that does not contain any bad triangles. Let A' be obtained by replacing a crossing of A by a vertex. If the vertex closure A' is a θ_4 graph, then A' contains a knot and hence is not a ravel.

Here is a counterexample when A contains a bad triangle:



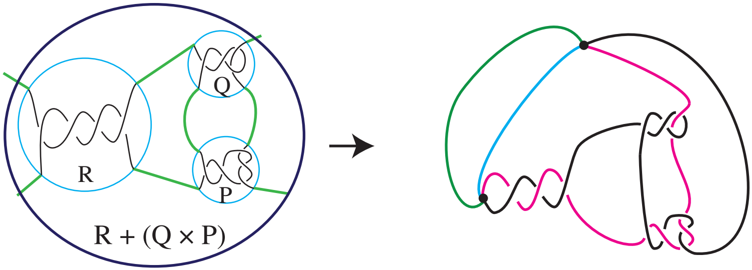
Counterexample with Bad Triangle





- θ_4 graph that contains no knots, but is non-planar, so it is a ravel.

Counterexample with Bad Triangle



- θ_4 graph that contains no knots, but is non-planar, so it is a ravel.
- Since it contains a non-planar θ_3, it is not almost unknotted.

Outline of Proof

Proof of Theorem is by induction on the number n of rational tangles in A. Recall that A is written with a minimal number of rational tangles.

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Base Case:

- Since A is not rational, $n \neq 1$.
- Suppose $n = 2$. Then, $A = A_1 + A_2$ or $A = A_1 \times A_2$ for some rational tangles A_1 and A_2.

Outline of Proof

Proof of Theorem is by induction on the number n of rational tangles in A. Recall that A is written with a minimal number of rational tangles.

Base Case:

- Since A is not rational, $n \neq 1$.
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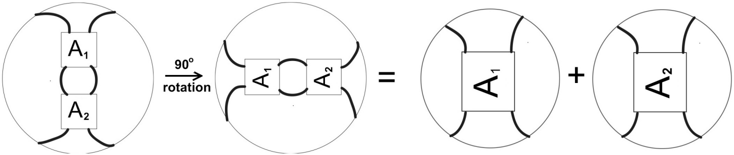
- $A_1 \times A_2$ can be expressed as a sum of two tangles:



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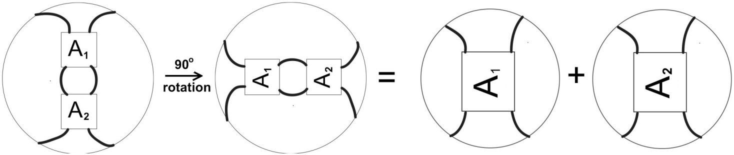

- Since we have already proved the result for Montesinos tangles, this completes the base case.

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Inductive Step:

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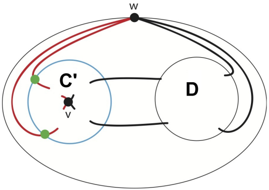
If C is not rational then, by the inductive hypothesis, the vertex closure of C' contains a knot. It follows that the vertex closure of A' will contain a knot.



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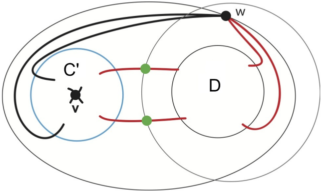
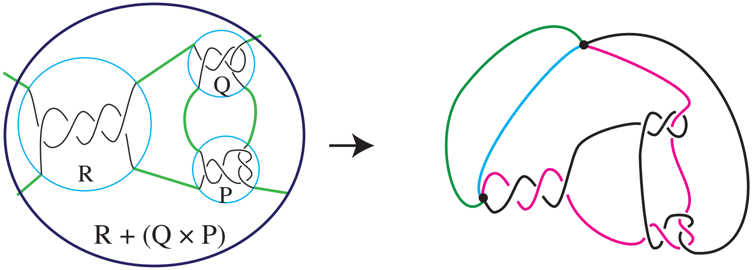


So the vertex closure of A' will also contain a knot. ∎

Future Research

Recall our counterexample:



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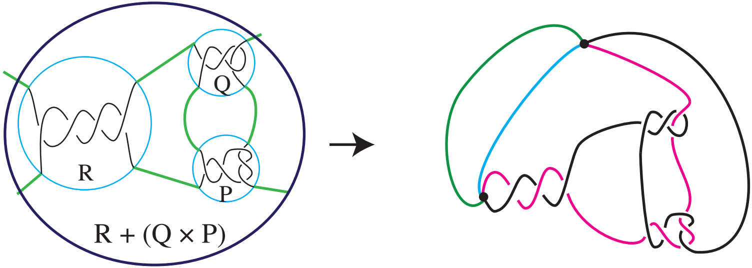

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If we obtain a 4-ravel from an algebraic tangle by replacing a crossing with a vertex and taking the vertex closure of the resulting graph, can it ever be almost unknotted?

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