GRAPHS OF 20 EDGES ARE 2–APEX, HENCE UNKNOTTED

Thomas W. Mattman

California State University, Chico

(Preprint available at www.arXiv.org)

Abstract:

A graph is 2–apex if it is planar after the deletion of at most two vertices. Such graphs are not intrinsically knotted, IK. We investigate the converse, does not IK imply 2–apex? We determine the simplest possible counterexample, a graph on nine vertices and 21 edges that is neither IK nor 2–apex. In the process, we show that every graph of 20 or fewer edges is 2–apex.

2–apex $\Longrightarrow$ not IK

Proposition: [BBFFHL,OT] A graph of the form $H * K_2$ is IK if and only if H is non–planar.

Corollary: If G is 2–apex, then G is not IK.

Idea: Then $G - a, b = H$ is planar and G is a subgraph of the non IK graph $H * K_2$.

Main Theorem: All graphs of 20 or fewer edges are 2–apex.

Corollary: An IK graph has at least 21 edges.

(Proved independently by Johnson, Kidwell, and Michael.)

Does non IK $\Longrightarrow$ 2–apex?

The converse is not true.

Example: $K_6 \sqcup K_6$ is neither IK nor 2-apex.

Simplest counterexample is a graph on nine vertices and 21 edges, E_9.

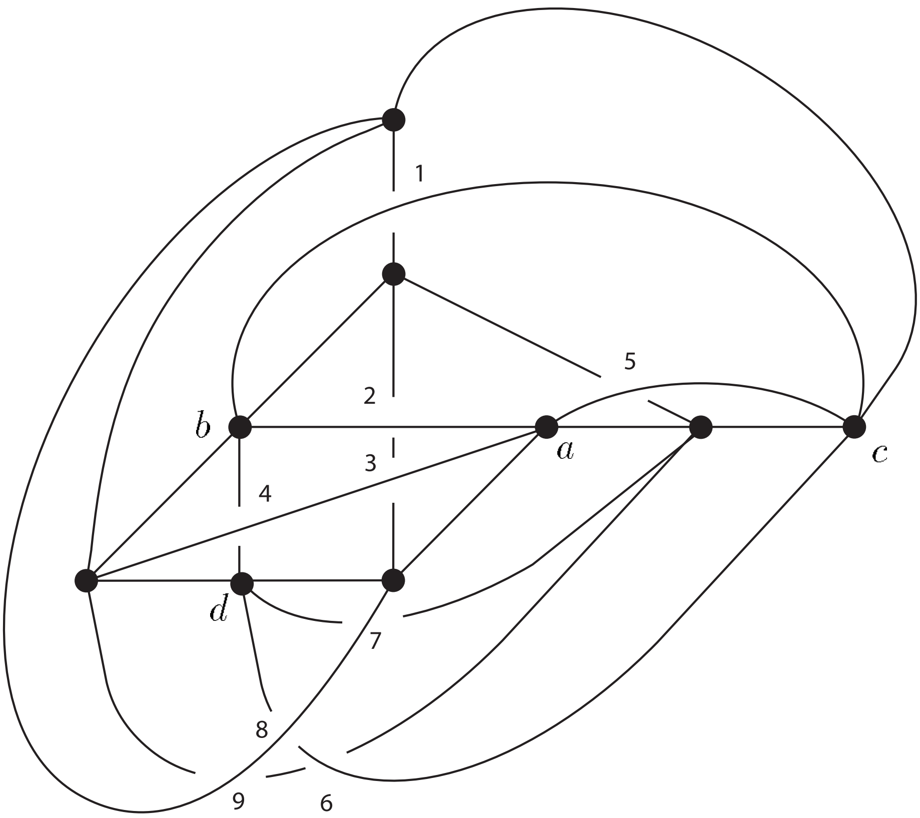

FIGURE 1. An unknotted embedding of E_9.

A Strategy for Classifying IK graphs.

But, exceptions may be rare, e.g., only 8 up to 9 vertices.

On the other hand, may be many minor minimal IK graphs.

This suggests a new classification strategy.

Problem 15: Classify the graphs that are neither IK nor 2–apex.

Then G is not IK if

1) G is 2-apex or

2) G is in the class of Problem 15.

Proof of Main Theorem

Let G be a graph of 20 edges, i.e., $\|G\| = 20$.

Six cases depending on the number of vertices, $|G|$.

1. $|G| \leq 8$, 2. $|G| = 9$, 3. $|G| = 10$,

4. $|G| = 11$, 5. $|G| = 12$, and 6. $|G| \geq 13$.

Today we'll look at the first and last case.

Strategy: Assume G is not 2–apex.

$\Longrightarrow \forall a, b \qquad G - a, b$ non-planar.

Using this assumption, show

$\exists G - a, b$ planar (contradiction)

Case 1: $|G| \leq 8$.

If $|G| \leq 7$, then G is a proper subgraph of K_7,

hence 2–apex.

So, assume $|G| = 8$.

Maximum Degree $\Delta(G)$

$\Delta(G) \leq 7$ (vertex has at most 7 neighbours).

On the other hand $\Delta(G) \geq 5$ since $4 \times 8 \leq 40$.

Note: minimum degree $\delta(G) \geq 3$.

Case 1: $|G| \leq 8$ and $\|G\| = 20$.

Then, $|G| = 8$, $\quad 5 \leq \Delta(G) \leq 7$, $\quad$ and $\delta(G) \geq 3$

Claim: $\exists a, b \ni \|G - a, b\| \leq 10$

Proof: Let a be a vertex of maximal degree.

If $d(a) = 7$, then,

$$7 + 3 \times 7 = 28 < 40 \Longrightarrow \exists b \text{ with } d(b) \geq 4.$$

If $d(a) = 6$, then,

$$6 + 4 \times 7 = 34 < 40 \Longrightarrow \exists b \text{ with } d(b) \geq 5.$$

If $d(a) = 5$, then, $\forall v, d(v) = 5$,

Choose b so that it's not adjacent to a.

(Here, $\|G - a, b\| = 10$.)

Case 1: $|G| \leq 8$ and $\|G\| = 20$.

Then, $|G| = 8, \qquad 5 \leq \Delta(G) \leq 7, \qquad$ and $\delta(G) \geq 3$

and $\exists a, b \ni \|G - a, b\| \leq 10$

Then $G - a, b = K_{3,3}$ or $K_{3,3} \cup v_1 v_2$.

<u>Subcase i</u>: $G - a, b = K_{3,3}$

Let $V(K_{3,3}) = \{v_1, v_2, v_3, w_1, w_2, w_3\}$.

We can assume 7 or $6 = d(a) \geq d(b)$.

Then $\|N(a) \cap N(b) \cap V(G - a, b)\| \geq 4$.

(Indeed, if $d(a) = 7$, then $d(b) = 5$.

If $d(a) = 6$, then $d(b) = 5$ or 6.)

So, we can assume v_1, v_2 are in intersection.

Case 1: $|G| \leq 8$ and $\|G\| = 20$.

<u>Subcase i</u>: $G - a, b = K_{3,3}$ and $|G| = 8$.

We can assume 6 or $7 = d(a) \geq d(b)$ and

$\{v_1, v_2\} \in N(a) \cap N(b) \cap V(G - a, b)$.

If $d(a) = 7$, then $G - a, v_1$ is planar.

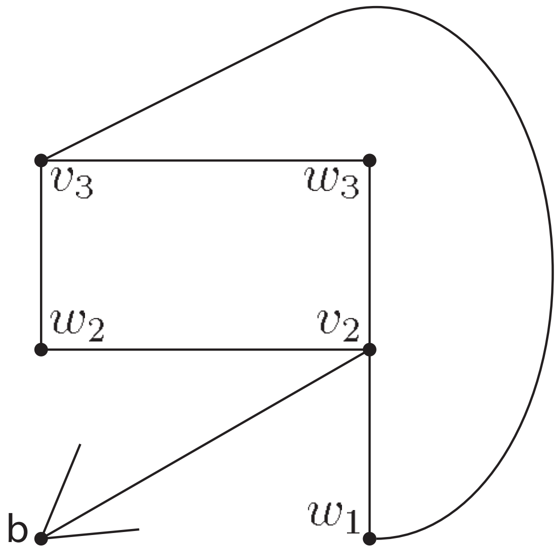

FIGURE 2. $G - a, v_1$

If $d(a) = 6$, then

$G - a, v_1$ not planar $\Longrightarrow \{w_1, w_2, w_3\} \subset N(b)$

(Same as figure above with one extra edge from b.)

Case 1: $|G| \leq 8$ and $\|G\| = 20$.

Subcase i: $G - a, b = K_{3,3}$ and $|G| = 8$.

We can assume $6 = d(a) \geq d(b) = 6$ or 5,

$\{v_1, v_2\} \in N(a) \cap N(b) \cap V(G - a, b)$,

and $\{w_1, w_2, w_3\} \in N(b)$.

Then $G - a, w_1$ is planar.

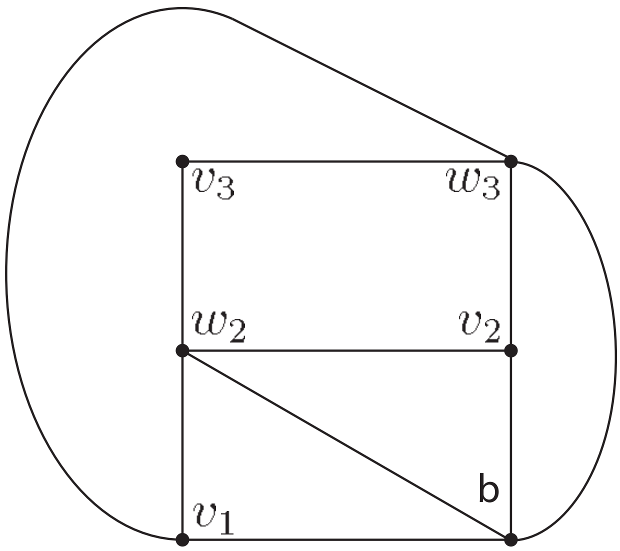

FIGURE 3. $G - a, w_1$

This completes subcase i.

Case 1: $|G| \leq 8$ and $\|G\| = 20$.

<u>Subcase ii</u>: $G - a, b = K_{3,3} \cup v_1 v_2$ and $|G| = 8$.

Here, $d(b) \leq d(a) \in \{5, 6, 7\}$.

If $d(a) = 5$, then $d(b) = 5$, too, and they're not adjacent.

Then, $\|N(a) \cap N(b)\|$ is 4 or 5.

If 5, then WLOG, $v_1 \in N(a) \cap N(b)$

$\Longrightarrow d(v_1) = 6$ (contradiction)

So, $N(a) \cap N(b) = \{v_3, w_1, w_2, w_3\}$.

Then $G - v_3, w_1$ is planar.

Case 1: $|G| \leq 8$ and $\|G\| = 20$.

Subcase ii: $G - a, b = K_{3,3} \cup v_1v_2$ and $|G| = 8$.

$d(b) \leq d(a) \in \{5, 6, 7\}$.

If $d(a) = 5$, then $G - v_3, w_1$ is planar.

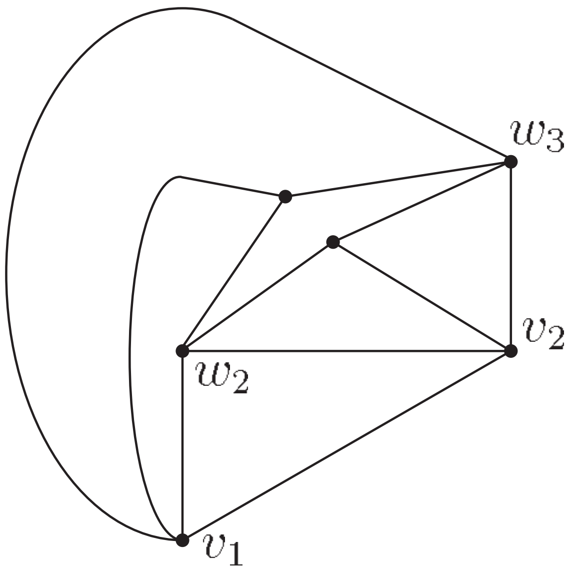

FIGURE 4. $G - v_3, w_1$ is planar. Unlabelled vertices are a and b.

So, we can assume $d(a) = 6$ or 7. Then $d(b) \leq 5$.

Case 1: $|G| \leq 8$ and $\|G\| = 20$.

<u>Subcase ii</u>: $G - a, b = K_{3,3} \cup v_1v_2$ and $|G| = 8$.

$d(b) \leq 5 < d(a) \leq 7$.

<u>Claim:</u> $\{w_1, w_2, w_3\} \subset N(b)$.

Otherwise, $G - a, v_1$ is planar:

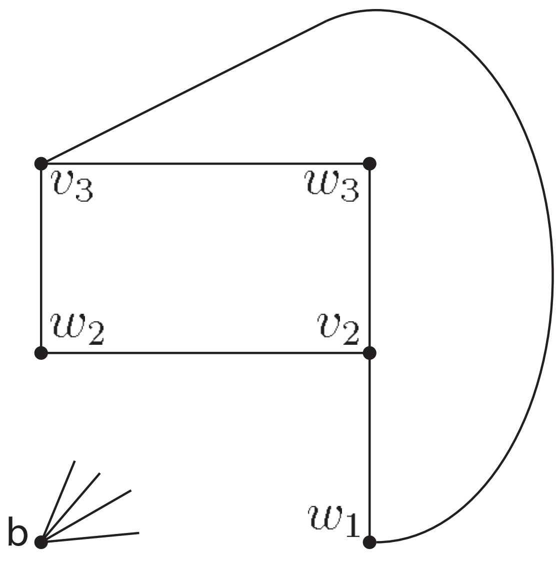

FIGURE 5. $G - a, v_1$

Case 1: $|G| \leq 8$ and $\|G\| = 20$.

Subcase ii: $G - a, b = K_{3,3} \cup v_1v_2$ and $|G| = 8$.

$d(b) \leq 5 < d(a) \leq 7$.

and $\{w_1, w_2, w_3\} \subset N(b)$.

Then $G - a, w_1$ is planar.

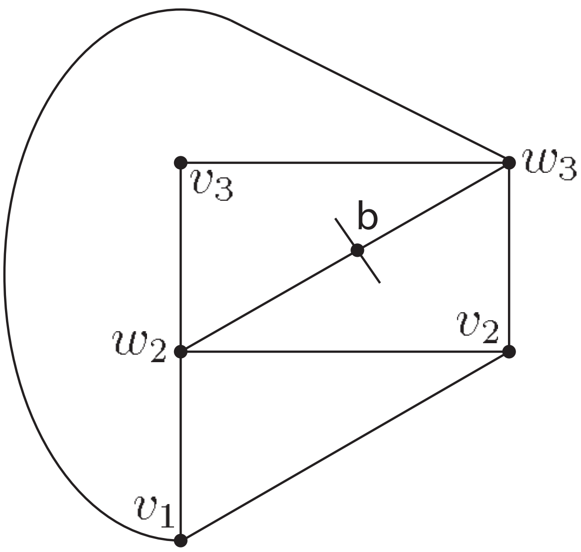

FIGURE 6. $G - a, w_1$

This completes proof of Case 1.

Case 6: $|G| \geq 13$ and $\|G\| = 20$.

If $|G| > 13$, then $\delta(G) \leq 3$ as

$$3 \times 14 = 42 > 40.$$

But then we can remove v with $d(v) \leq 2$

Induction $\Longrightarrow$ $G - v$ is 2-apex whence G is also.

So assume $|G| = 13$. Again $\delta(G) \geq 3$ implies

Degree Sequence $= \{4, 3, 3, \ldots, 3\}$.

Let $d(a) = 4$ and choose b not adjacent to a.

Then $\|G - a, b\| = 13$.

Case 6: $|G| \geq 13$ and $\|G\| = 20$.

We can assume $|G| = 13$, $d(a) = 4$, $d(b) = 3$,

and $\|G - a, b\| = 13$.

Also $\Delta(G - a, b) = 3$, and $\delta(G - a, b) \geq 1$.

Then $\chi(G - a, b) = 11 - 13 = -2$.

$G - a, b$ non-planar $\Longrightarrow$ component with K_5 or $K_{3,3}$ minor.

Now, $\chi(K_5) = 5 - 10 = -5$.

Taking minors cannot increase χ.

Non tree components have $\chi \leq 0$.

Since $\delta(G - a, b) \geq 1$, the only possibility is

$G - a, b = K_5 \sqcup K_2 \sqcup K_2 \sqcup K_2$.

However, this contradicts $\Delta(G - a, b) = 3$.

Case 6: $|G| \geq 13$ and $\|G\| = 20$.

$|G| = 13$, $\Delta(G - a, b) = 3$, $\delta(G - a, b) \geq 1$

and $\chi(G - a, b) = -2$.

So, $G - a, b$ has a component C_1 with $K_{3,3}$ minor.

$\chi(K_{3,3}) = 6 - 9 = -3$.

$\delta(G - a, b) \geq 1 \Longrightarrow$

At most two trees in addition to C_1.

So, $\chi(C_1) = -3$ or -4.

Case 6: $|G| \geq 13$ and $\|G\| = 20$.

<u>Subcase i</u> $\chi(C_1) = -3$.

Here $\exists! T$ (tree) with $2 \leq |T| \leq 5$.

If $|T| = 2$, then $C_1 = G - a, b \setminus T$ is non planar

$|C_1| = 9$ and $\|C_1\| = 12$.

Since $\Delta(C_1) = 3$ and $\delta(C_1) \geq 1$, $\exists v$ with $d(v) = 2$.

Then C_1 is $K_{3,3} \cup C_3$

or a graph below $+\ v$.

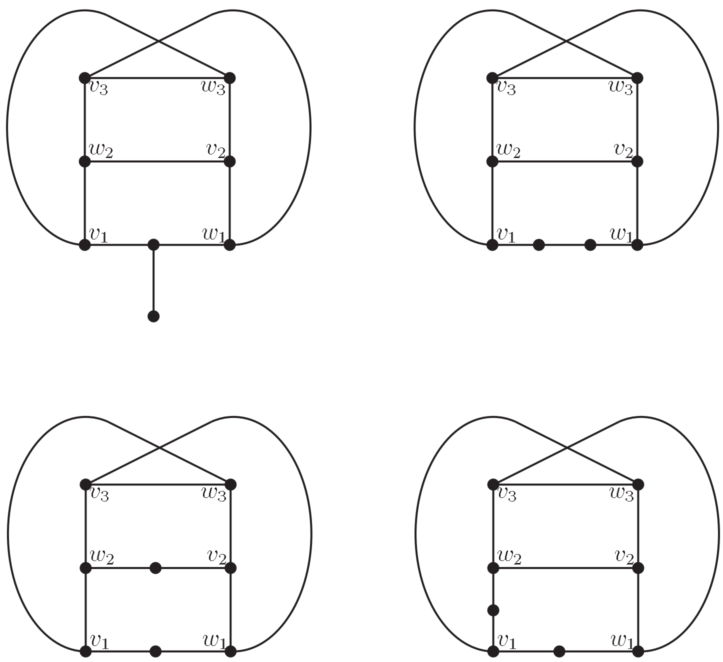

FIGURE 7. Obtain C_1 by adding v with $d(v) = 2$.

Case 6: $|G| \geq 13$ and $\|G\| = 20$.

Subcase i $\chi(C_1) = -3$.

Assume $\exists! T$ with $|T| = 2$, i.e., $T = K_2$.

Then $C_1 = G - a, b - T$ is $K_{3,3} \sqcup C_3$

or one of the graphs above $+ v$.

If $G - a, b = K_2 \sqcup K_{3,3} \sqcup C_3$,

then $G - a, c$ is planar where $c \in V(K_{3,3})$.

(Since a is the unique vertex of degree 4

in G, $N(b) \cap V(K_{3,3}) = \emptyset$.)

On the other hand if C_1 is as in figure above,

$G - a, v_1$ is planar.

This completes case where $|T| = 2$.

Case 6: $|G| \geq 13$ and $\|G\| = 20$.

Subcase i $\chi(C_1) = -3$.

Assume $\exists! T$ with $|T| = 3$

Then $C_1 = G - a, b \setminus T$ is one of four graphs:

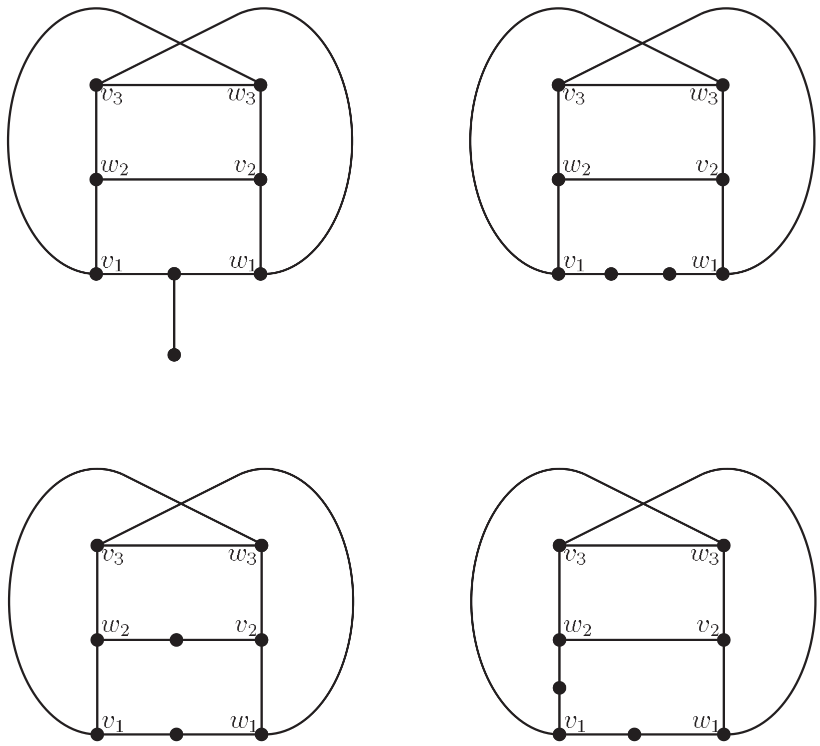

FIGURE 8. C_1 is one of these four.

In any case $G - a, v_1$ is planar.

This completes case where $|T| = 3$.

Case 6: $|G| \geq 13$ and $\|G\| = 20$.

<u>Subcase i</u> $\chi(C_1) = -3$.

Assume $\exists! T$ with $|T| = 4$

Here $C_1 = G - a, b \setminus T$ must be the graph

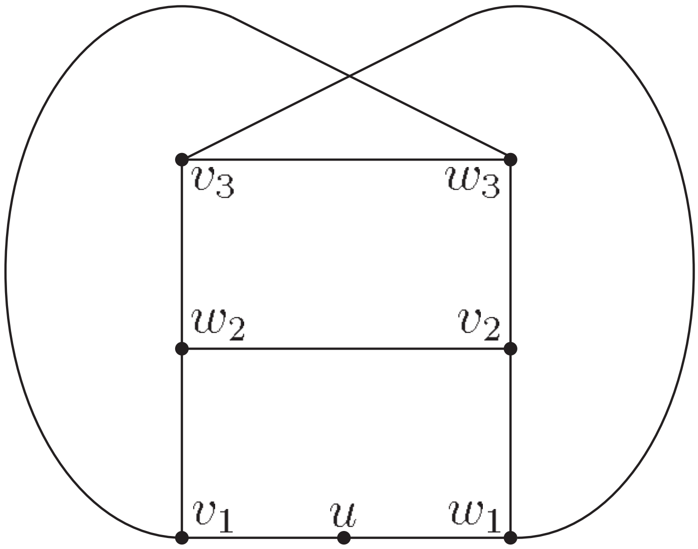

FIGURE 9. C_1 in case $|T| = 4$.

Again, $G - a, v_1$ is planar.

Similarly, if $|T| = 5$, then $C_1 = K_{3,3}$

and $G - a, v_1$ is planar.

This completes Subcase i.

Case 6: $|G| \geq 13$ and $\|G\| = 20$.

<u>Subcase ii</u> $\chi(C_1) = -4$.

$\exists T_1$ and T_2.

But $\delta(G - a, b) \geq 1 \Longrightarrow T_1 = K_2$ and $|T_2| = 2$ or 3.

If $|T_2| = 2$, then $T_2 = K_2$, too

and $C_1 = G - a, b \setminus (T_1 \sqcup T_2)$ has

$|C_1| = 7$ and $|C_1| = 11$.

Then C_1 is

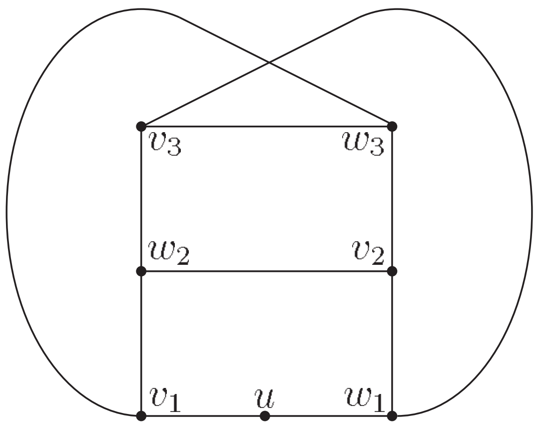

FIGURE 10. C_1 in case $|T| = 4$.

whence $G - a, v_1$ is planar.

Case 6: $|G| \geq 13$ and $\|G\| = 20$.

Subcase ii $\chi(C_1) = -4$.

$\exists T_1$ and T_2 with $T_1 = K_2$ and $|T_2| = 2$ or 3.

If $|T_2| = 3$, then $C_1 = G - a, b \setminus (T_1 \sqcup T_2)$

has $|C_1| = 6$ and $\|C_1\| = 10$.

So, $C_1 = K_{3,3} \sqcup v_1 v_2$.

This is a contradiction as a is the unique vertex of degree 4.

This completes the argument in Case 6.