An algorithm for detecting intrinsically knotted graphs, yielding many new minor minimal IK graphs

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Definition:
A graph G is **intrinsically knotted** (IK) if every embedding of G in S^3 contains a nontrivial knot.

Theorem (Conway & Gordon, 1983):
K_7 is IK.

Main question: Which graphs are IK?

Definition:
H is a **minor** of G if H can be obtained from a subgraph of G by contracting edges.

Definition:
G is **minor minimal intrinsically knotted** (MMIK) if G is IK but no minor of G is IK.

Example: K_7 is MMIK: if we delete or contract any edge, it will have a "knotless" embedding (i.e., no nontrivial knots).

Fact: A graph is IK iff it has a MMIK minor. (proof is easy)

The Graph Minor Theorem (Robertson, Seymour, 1986-97: 20 papers, 500 pages) In every infinite set of graphs, at least one is a minor of another.

Corollary: Given *any* property, there are finitely many minor minimal graphs with that property.

$\Rightarrow$ Finitely many MMIK graphs

Corollary: IKness is decidable. (But we don't know how!)

Proof:

Let $S =$ finite set of MMIK graphs.

Given any graph G, check whether there is a graph in S that is a minor of G.

Known MMIK graphs

- K_7 (Conway & Gordon, 1983)

- $K_{3,3,1,1}$ (Foisy, 2000)

- Graphs obtained from K_7 and $K_{3,3,1,1}$ by ∇Y moves (Kohara & Suzuki, 1990)

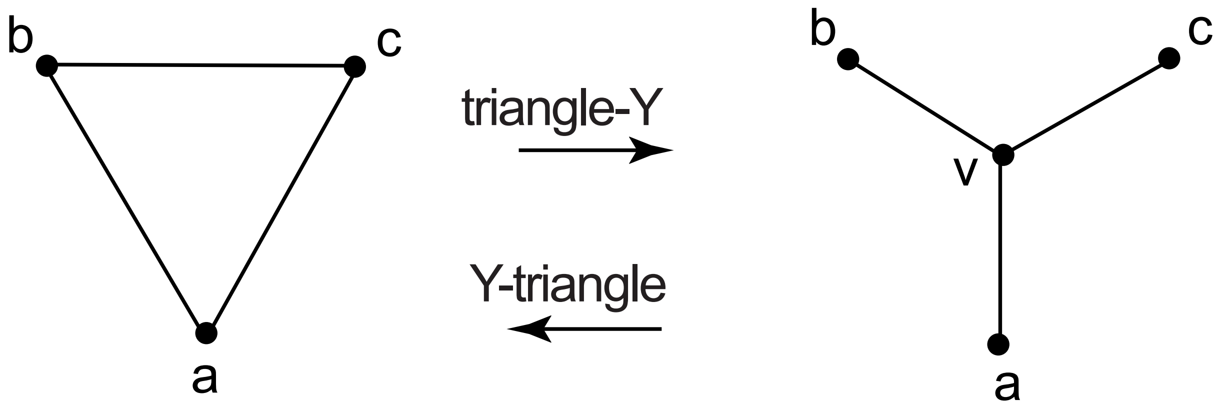
- $F_{13,30}$ (Foisy, 2002)

Total: 41 graphs.

Foisy (2006) found 4 more IK graphs. It is not known if they are MM. But he showed they don't contain any of the above 41 as minors.

Goldberg, Mattman, & N. (2009) : 222 new MMIK graphs.

Definition: ∇Y **move**:



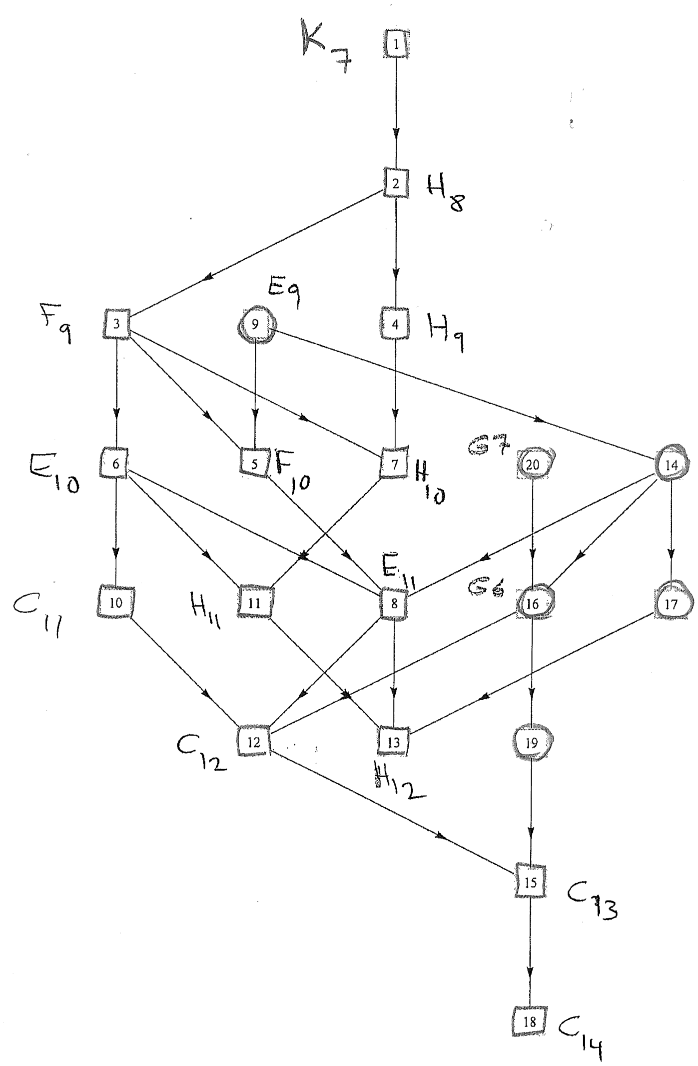
Terminology:

Parent $\xrightarrow{\nabla \mathrm{Y}}$ **Child**

Ancestor $\xrightarrow{\nabla \mathrm{Y}} \cdots \xrightarrow{\nabla \mathrm{Y}}$ **Descendant**

Cousin $\xrightarrow{M_1} \cdots \xrightarrow{M_n}$ **Cousin**, $M_i = \nabla\mathrm{Y}$ or $\mathrm{Y}\nabla$, $n \geq 0$.

Theorem (Sachs 1983, Motwani, Raghunathan, Saran 1988): If G is IK, its descendants are also IK.



□ : MMIK [KS]

○ : not IK [GMN]

The K_7 family consists of 20 graphs:

14 graphs: K_7 and its descendants already known to be MMIK.

6 graphs: cousins but not descendants of K_7 are not IK.

(Hanaki, Nikkuni, Taniyama, Yamazaki, arXiv. Goldberg, Mattman, N., still writing up!)

New MMIK graphs

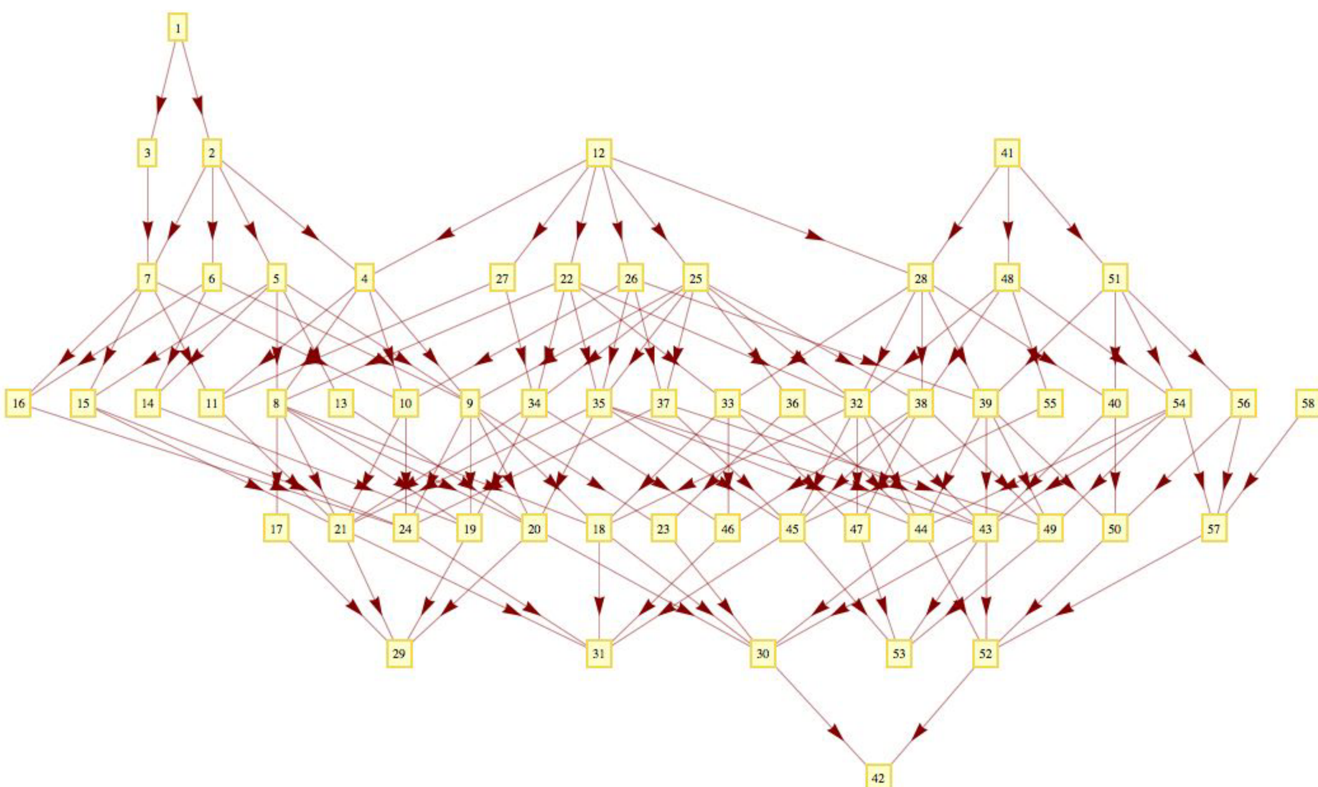
— $K_{3,3,1,1}$ family:

58 graphs: all are MMIK!

(This answers Question 3.3 of the problem list)

26 graphs: $K_{3,3,1,1}$ and its descendants were already known to be MMIK

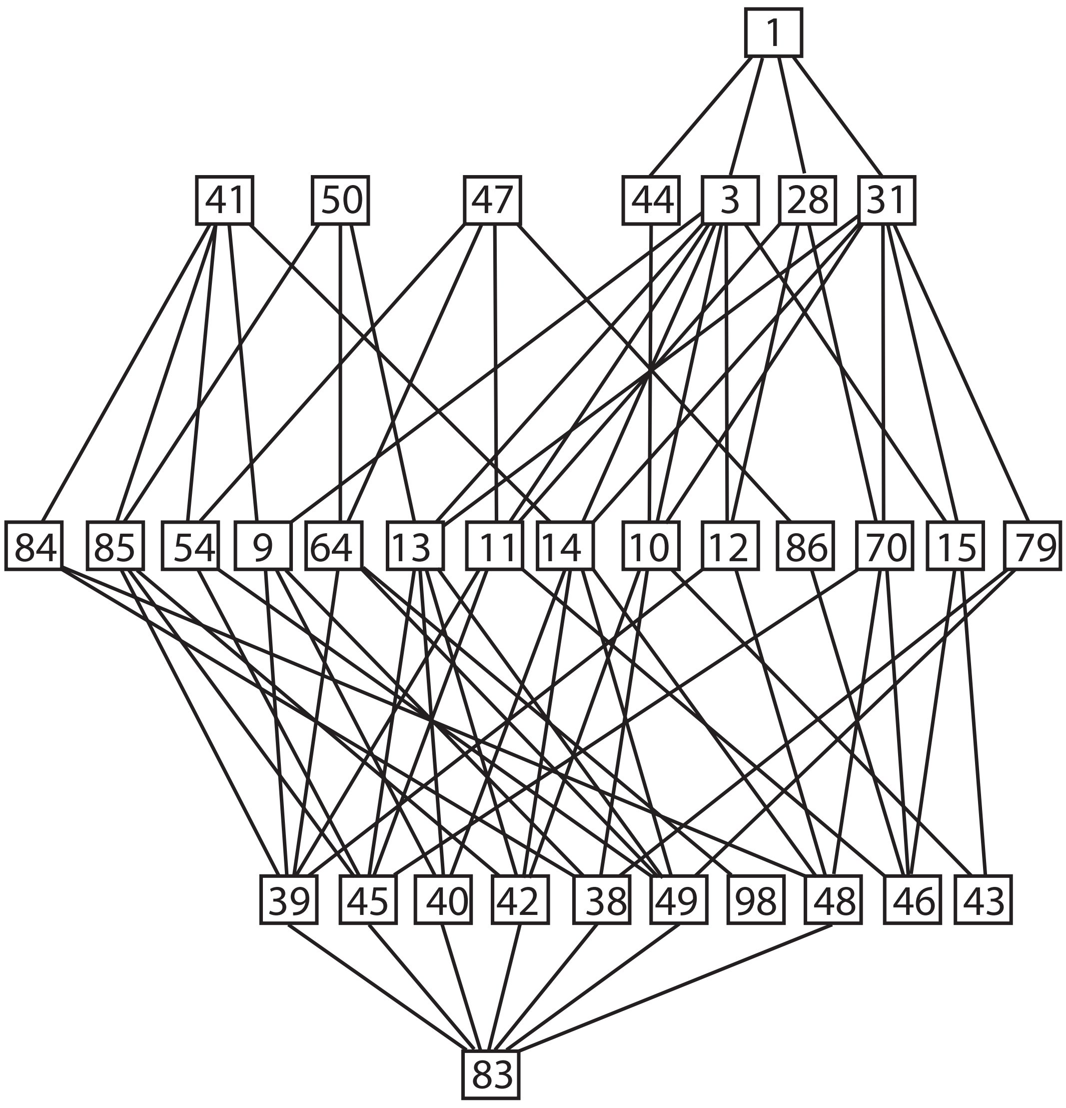
$58 - 26 = 32$ new MMIK graphs



— $E_9 + e$ family:

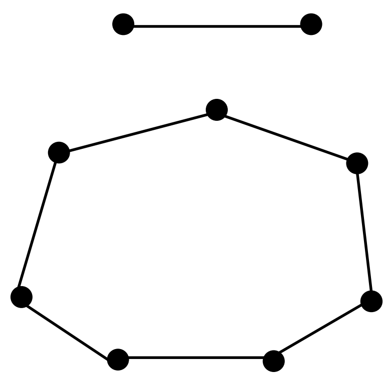
110 cousins; all are IK.

Only 33 are MMIK.



— G_{28} family:

Complement of $G_{28} =$ 7-cycle + one edge :



1601 cousins, all IK.

We verified 156 as MMIK.

Out of 461 tested, the program reported that *probably*: 458 MMIK, 3 not MMIK.

— Monster family:

Over 600,000 cousins (we don't know exactly how many)

Obtained from an IK minor of a non-IK descendant of G_{28},
G28C1413SM6SM14SM40 (25 edges):

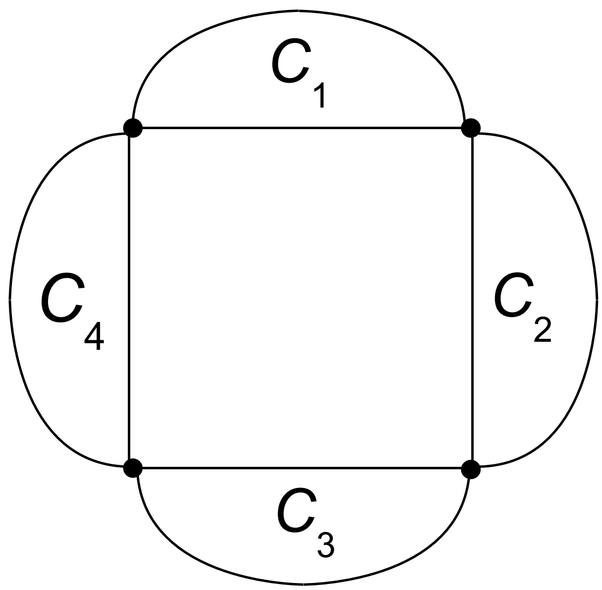
G28C1413SM6SM14SM40 itself is MMIK. But it seems most of its cousins are not IK.

New MMIK graphs: $32 + 33 + 156 + 1 = 222$

Is it worthwhile to search for more?

(I think there are probably thousands more!)

D_4 graph:



Definition: A spatial D_4 is **double linked** if $\mathrm{lk}(C_1, C_3) \neq 0$, and $\mathrm{lk}(C_2, C_4) \neq 0$.

D_4 Lemma:

[Taniyama & Yasuhara, 2001. Foisy(mod2) 2001]:

If a spatial graph G contains a double linked D_4 as a minor, then G contains a nontrivial knot.

Corollary:

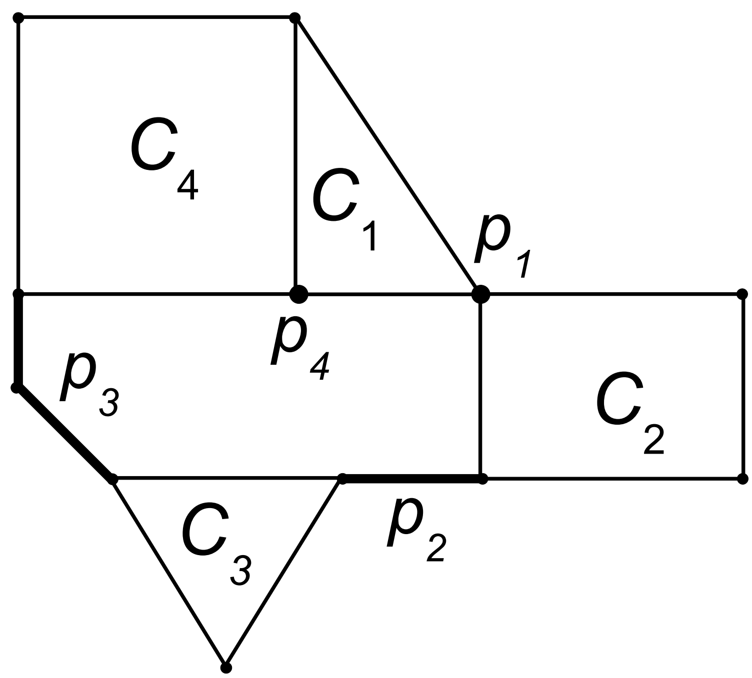
If every embedding of G has a double linked D_4 as a minor, then G is IK.

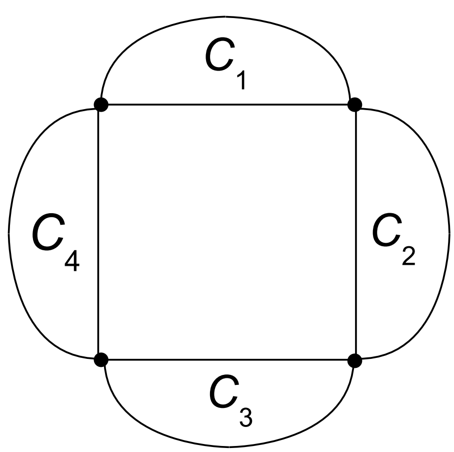
Definition: If an embedding of a graph contains no double linked D_4 as a minor, we call it a **$\mathbf{D_4}$-less embedding**.

Our computer program searches for D_4-less embeddings.

Outline of algorithm:

1. Find all **quads**, i.e., sets of cycles (C_1, C_2, C_3, C_4) that can "produce" a D_4.





2. For each pair of disjoint edges e_i, e_j, let x_{ij} denote the total number of signed crossings between e_i and e_j.

3. For each pair of disjoint cycles C, C', write $\mathrm{lk}(C, C')$ in terms of the variables x_{ij}.

4. For each quad (C_1, C_2, C_3, C_4), let

$$\mathrm{lk}(C_1, C_3) = 0 \text{ or } \mathrm{lk}(C_2, C_4) = 0$$

5. Solve systems of linear equations.

- If there are Q quads, there are 2^Q systems of equations.

- If none of the systems has a solution, then there is no D_4-less embedding; so G is IK.

- If one of the systems has a solution, the solution may help us find a knotless embedding for G.

Question 1:

G has a D_4-less embedding $\Rightarrow$
G has a knotless embedding?

If true, then our computer program decides IKness.

Question 2:

G has a D_4-less embedding $\Rightarrow$
G has an Arfless embedding?

(i.e., every cycle has Arf $= 0$)

Question 3:

G has an Arfless embedding $\Rightarrow$
G has a knotless embedding?

For all of the above, "$\Leftarrow$" is true.

Have written a similar program for deciding if a graph is intrinsically linked (IL).

The program relies on a result of [RST] (Robertson, Seymour, Thomas):

G is IL $\iff$ G is "IL mod 2".

Note: By [RST], we can check if a graph is IL by simply checking if it contains a Petersen Family minor. I don't know if such a program has been written yet.

It might be possible to answer other questions using similar computer programs; e.g.,

Problem 2.2 of problem list: Does every spatial embedding of $K_{4,4,4}$ contain two disjoint, non-split, 2-component links?

Useful lemma for proving MM:

Lemma [GMN]: If G is IK and has a MMIK child, then G is MMIK.

Example: In $K_{3,3,1,1}$ family, first prove the orphans (cousins 1, 12, 41, 58) are IK; then prove the infertiles (cousins 29, 31, 42, 53) are MM.