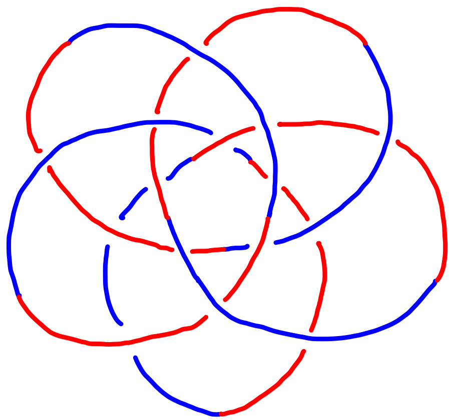
Flowers of knots

—— LR number and bridge index of knots ——

研究集会「ハンドル体結び目とその周辺 18」

2025年10月26日　東京理科大学 神楽坂キャンパス



谷山 公規　(早稲田大学 教育学部)

OU sequence of knot diagrams

R. Higa, Y. Nakanishi, S. Satoh and T. Yamamoto, Crossing information and warping polynomials about the trefoil knot, Journal of Knot Theory and Its Ramifications **21**, No. 12 (2012).

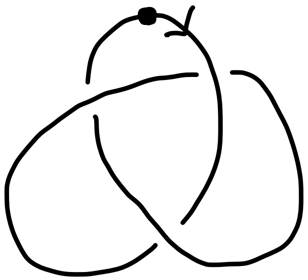
O (over) U (under)

LR sequence of knot shadows

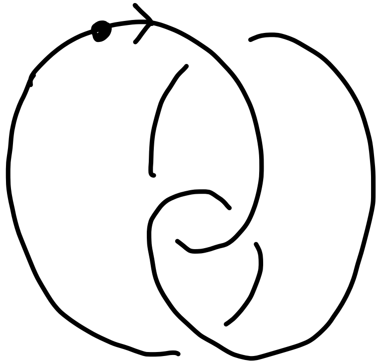
K. Takaoka, LR Number of spherical closed curves, *Tokyo J. Math.*, **38** (2015), 491-503.

L (left) R (right)

例



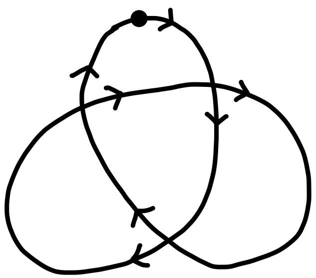
D_1



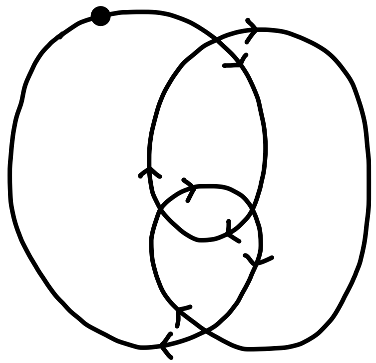
D_2

$$W^{OU}(D_1) = O\,U\,O\,U\,O\,U = (OU)^3$$

$$W^{OU}(D_2) = O\,O\,U\,U\,O\,O\,U\,U = O^2U^2O^2U^2 = (O^2U^2)^2$$



P_1



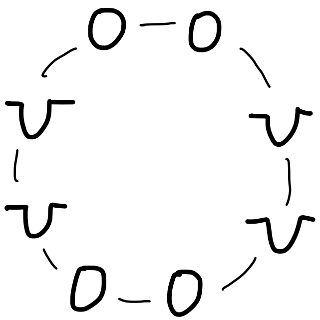
P_2

$$W^{LR}(P_1) = R\,L\,R\,L\,R\,L = (RL)^3$$

$$W^{LR}(P_2) = R\,R\,L\,L\,R\,R\,L\,L = R^2L^2R^2L^2 = (R^2L^2)^2$$

◎基点のとり方によらない巡回語として考える。

$$OOUUOO\underline{U}\,\underline{U} = \underline{U}OOUUOO\underline{U} = \underline{U}\,\underline{U}OOUUOO$$

$=$ 

これを **OU word** と呼ぶ。

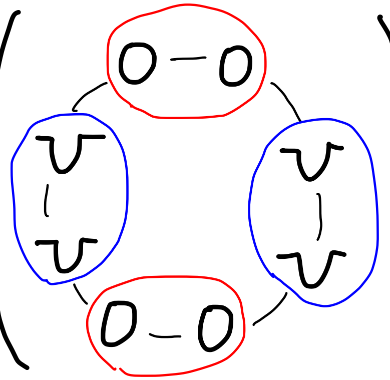
任意の OU word は $O^{a_1}U^{b_1}O^{a_2}U^{b_2}\cdots O^{a_k}U^{b_k}$ の形で表せる。

ここで $a_1+a_2+\cdots+a_k = b_1+b_2+\cdots+b_k = c(D)$

$ou(O^{a_1}U^{b_1}O^{a_2}U^{b_2}\cdots O^{a_k}U^{b_k}) := k$

$ou(\phi) := 1$

例

$ou\Bigg($  $\Bigg) = 2$

同様に LR sequence も巡回語として考える。

これを LR word と呼ぶ。

任意の LR word は $L^{a_1}R^{b_1}L^{a_2}R^{b_2}\cdots L^{a_k}R^{b_k}$ の形で表せる。

ここで $a_1+a_2+\cdots+a_k = b_1+b_2+\cdots+b_k = c(P)$

$\ell r(L^{a_1}R^{b_1}L^{a_2}R^{b_2}\cdots L^{a_k}R^{b_k}) := k$

$\ell r(\phi) := 1$

K : knot

$$ou(K) := \min\{ou(W^{OU}(D)) \mid D : \text{diagram of } K\}$$

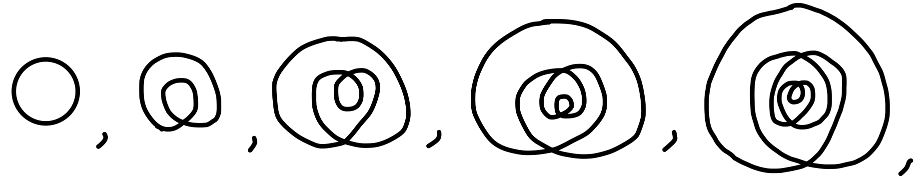
$$\boxed{ou(K) = bridge(K)}$$

$$lr(K) := \min\{lr(W^{LR}(P)) \mid P : \text{shadow of } K\}$$

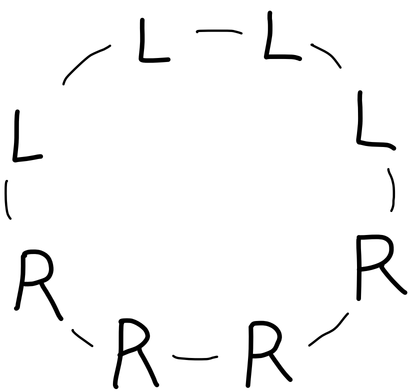
LR number of K

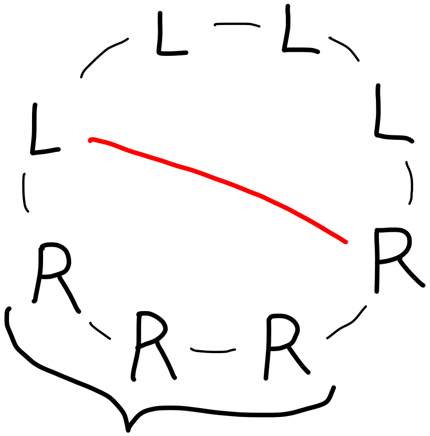
$$lr(K) = ?$$

Prop. [Takaoka]

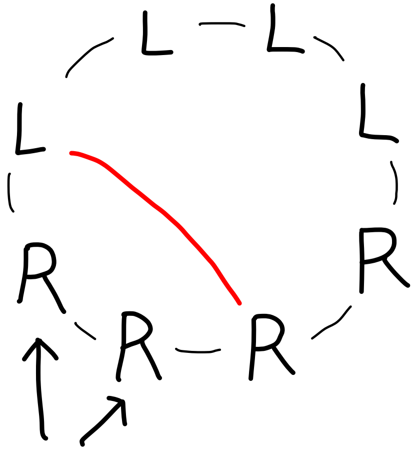
$$lr(W^{LR}(P)) = 1 \iff P \in \{$$  $$, \ldots \}$$

up to $\approx$ on $S^2 = \mathbb{R}^2 \cup \{\infty\}$



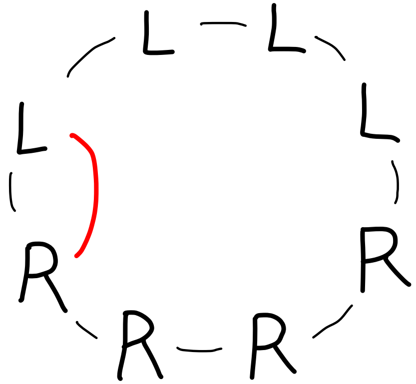


奇数はなし

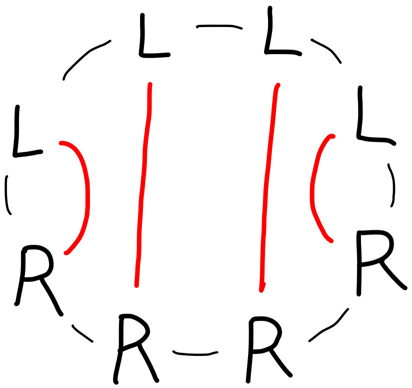


無理

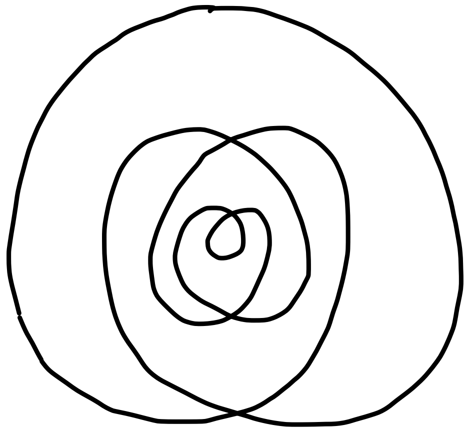
よって



これを続けて



これの実現は



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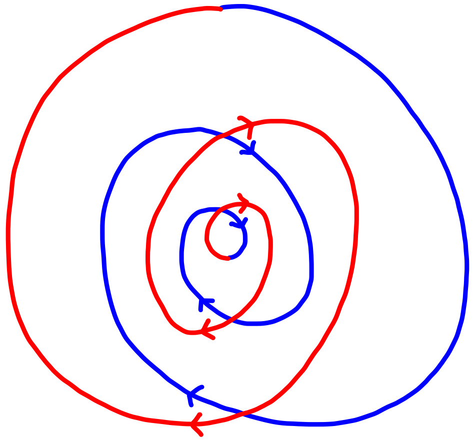
Cor. K : knot

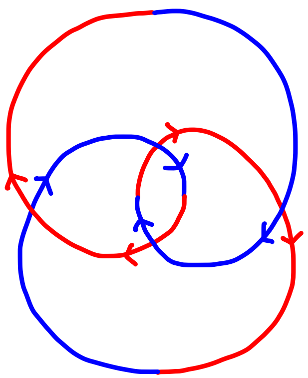
$\ell r(K) = 1 \Leftrightarrow K$: trivial

Theorem K : knot, $\ell r(K) = \mathrm{bridge}(K)$

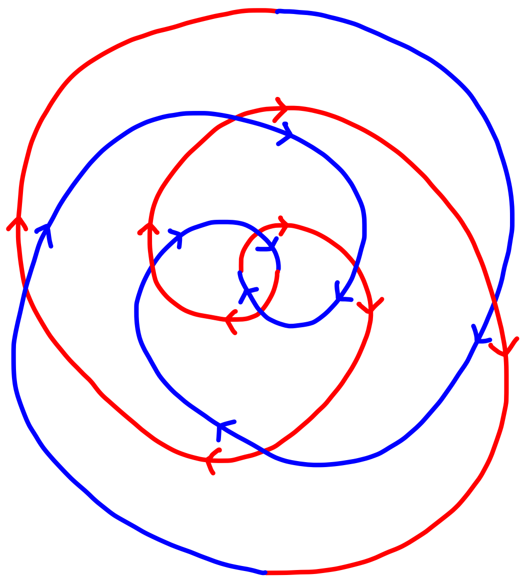


$LLLLRRRR = L^4R^4$

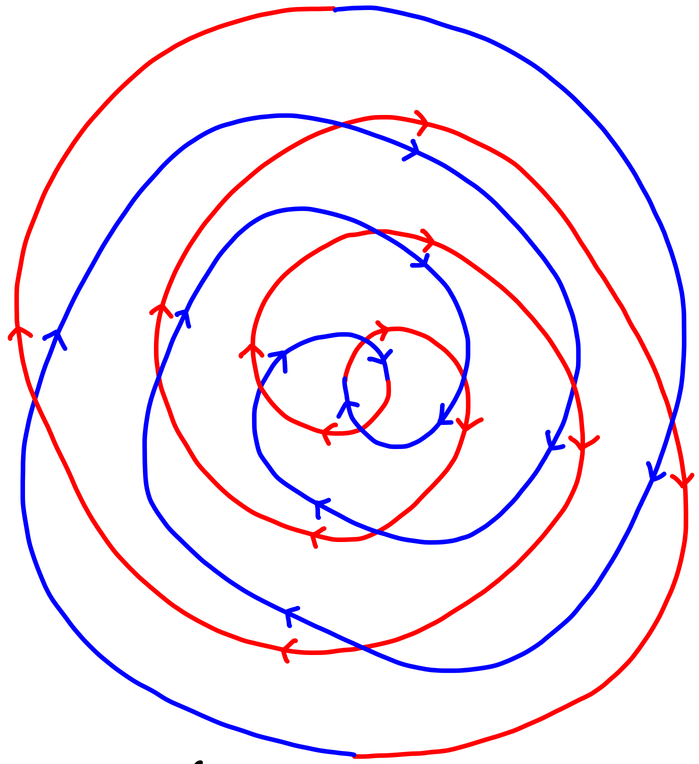
例 ($\ell r = 2$ の knot shadow の例)



$(L^2R^2)^2$



$(L^3R^3)^2$

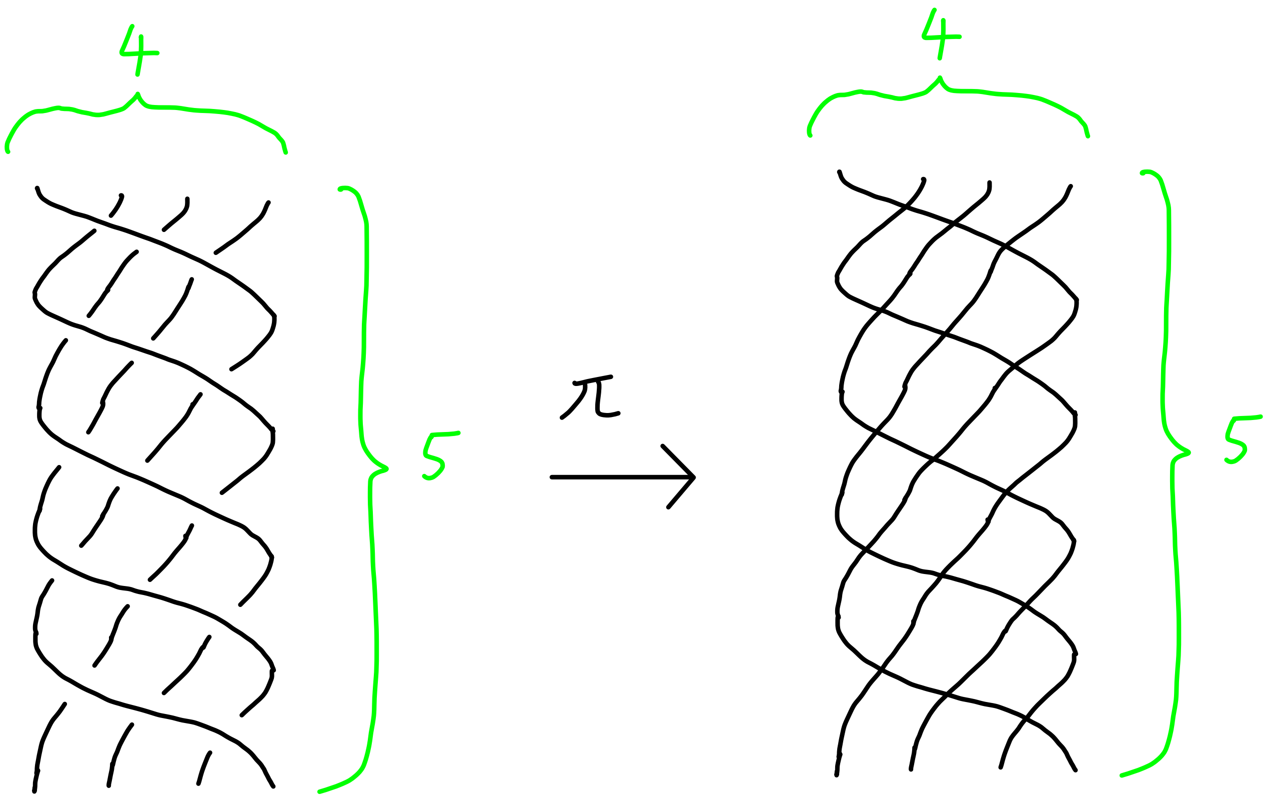


$(L^4R^4)^2$

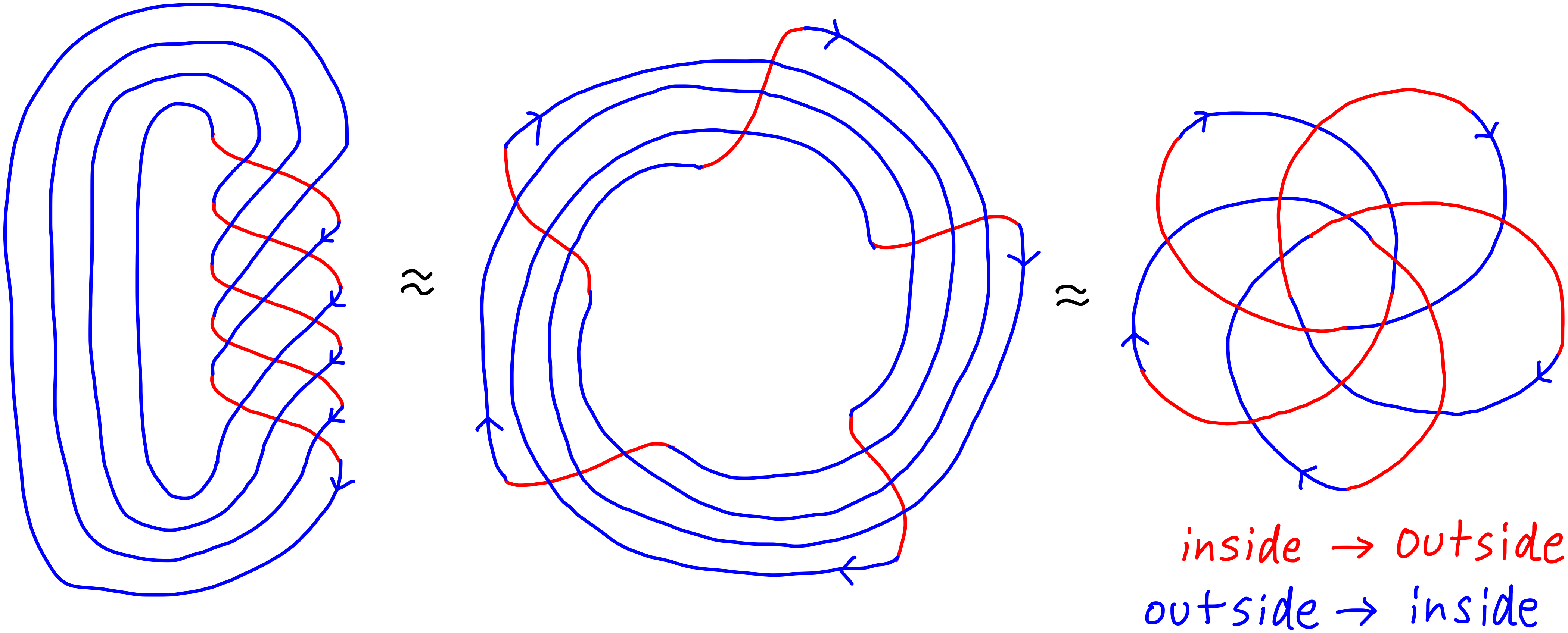
$n, k \in \mathbb{N}$, $\quad \pi : \mathbb{R}^3 \to \mathbb{R}^2$, $\quad \pi(x, y, z) = (x, y)$

$$F(n, k) := \pi\left(\mathrm{cl}\left((\sigma_1 \sigma_2 \cdots \sigma_{n-1})^k\right)\right)$$
$$= \mathrm{cl}\left(\pi\left((\sigma_1 \sigma_2 \cdots \sigma_{n-1})^k\right)\right)$$

Ex. F(4,5)



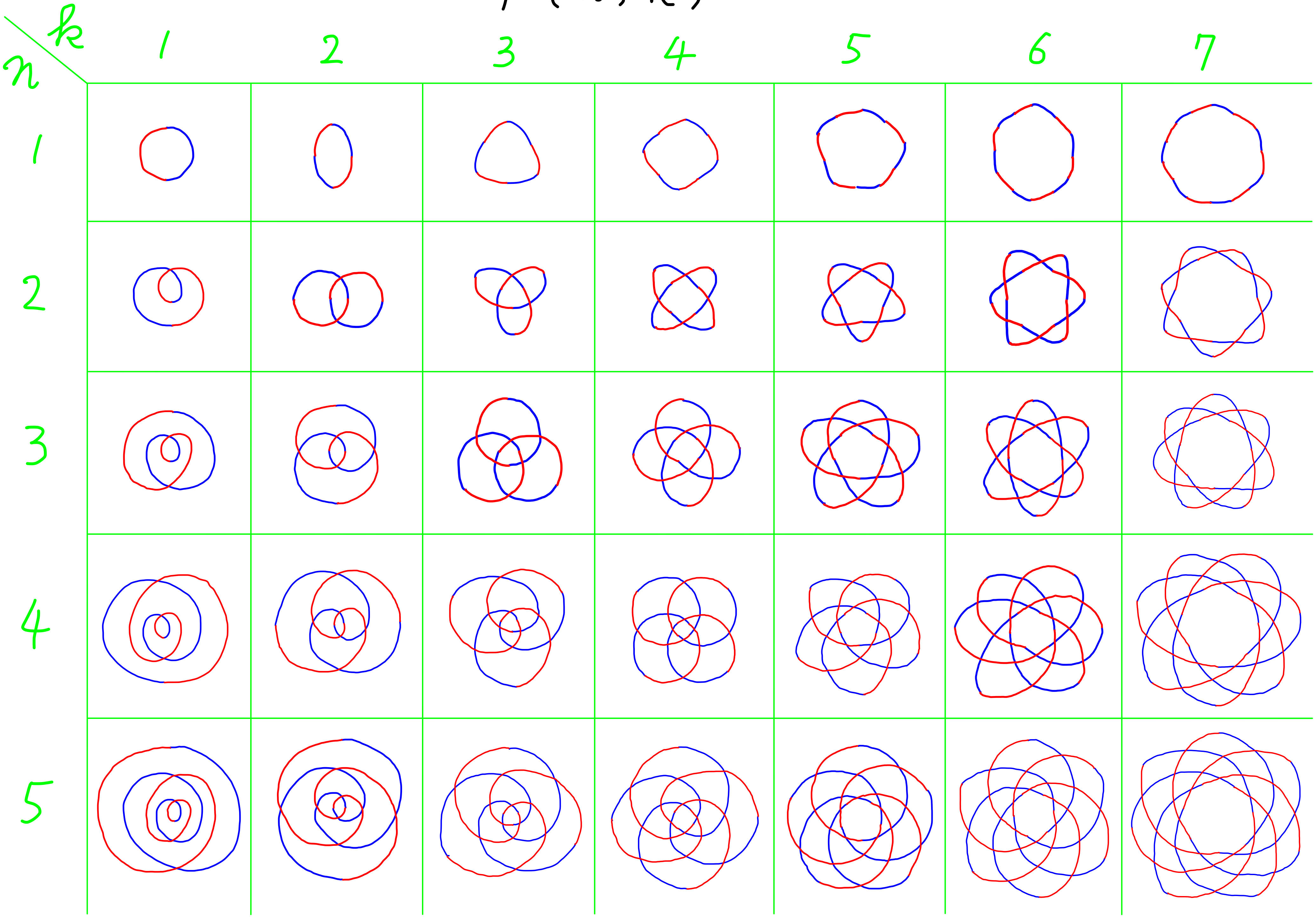
$(\sigma_1 \sigma_2 \sigma_3)^5$ $\qquad$ $\pi\left((\sigma_1 \sigma_2 \sigma_3)^5\right)$



$$\mathrm{cl}\left(\pi\left((\sigma_1\sigma_2\sigma_3)^5\right)\right) = F(4,5)$$

(n,k)-flower $F(n,k)$ is the shadow of the standard diagram of a (n,k)-torus knot

$F(n, k)$

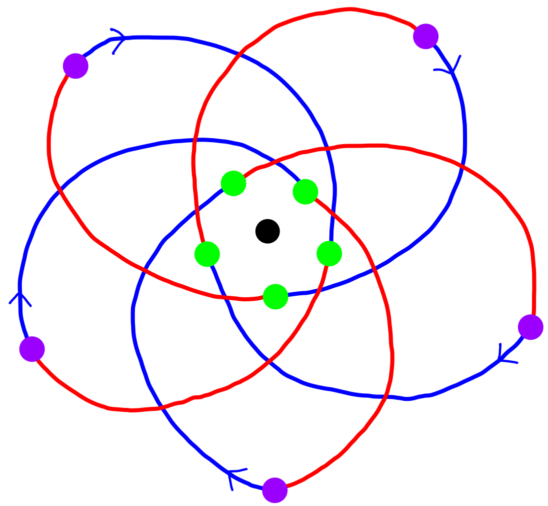


K : knot, $\mathrm{PROJ}(K)$: the set of all shadows of K

Prop. $F(n,k) \in \mathrm{PROJ}(K)$

$\Rightarrow$ (1) $\mathrm{braid}(K) \leqq n$

(2) $\mathrm{bridge}(K) \leqq k$



inside → outside = descending

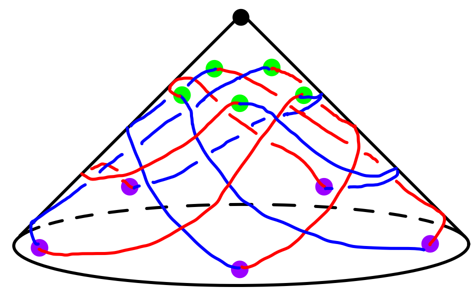
outside → inside = ascending

Proof (1) : Clear.

(2) Put $F(n,k)$ on a cone

Then k local maximums

and k local minimums



crossing over/under can be chosen arbitrary

C. Lamm, Zylinder-Knoten und symmetrische Vereinigungen, Ph.D. thesis, University of Bonn, Bonner Mathematische Schriften No. 321 (1999).
C. Lamm, Fourier knots, arXiv:1210.4543, [English translation of a part of "Zylinder-Knoten und symmetrische Vereinigungen", Ph.D. thesis, University of Bonn, Bonner Mathematische Schriften No. 321 (1999)].
V. O. Manturov, A combinatorial representation of links by quasitoric braids, *European J. Combin.*, **23** (2002) 207-212.

<u>Thm</u> [Lamm 1999] [Manturov 2002]

K : knot, $n \geqq \mathrm{braid}(K)$

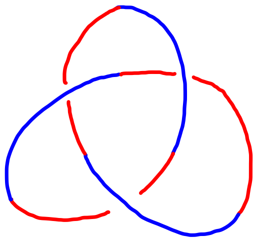
$\Rightarrow {}^{\exists}k \in \mathbb{N}$ s.t. $F(n,k) \in \mathrm{PROJ}(K)$

<u>Main theorem</u>

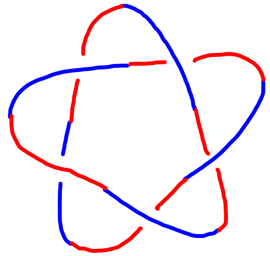
K : knot, $k \geqq \mathrm{bridge}(K)$

$\Rightarrow {}^{\exists}n \in \mathbb{N}$ s.t. $F(n,k) \in \mathrm{PROJ}(K)$

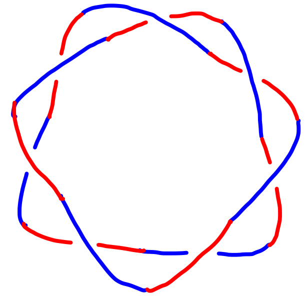
Ex. 3_1 $\quad$ $g.c.d.(n,k)=1,\ n,k \geqq 2 \Rightarrow F(n,k) \in PROJ(3_1)$



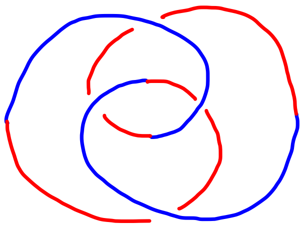
(2,3)



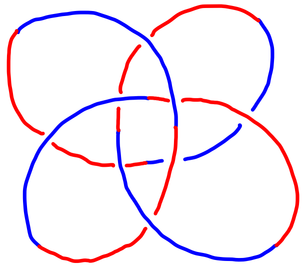
(2,5)



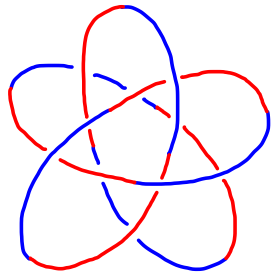
(2,7)



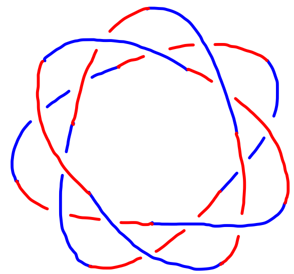
(3,2)



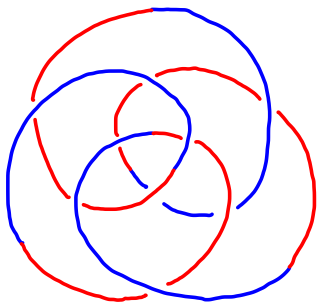
(3,4)



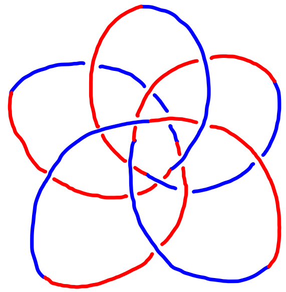
(3,5)



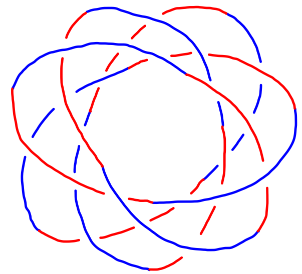
(3,7)



(4,3)



(4,5)

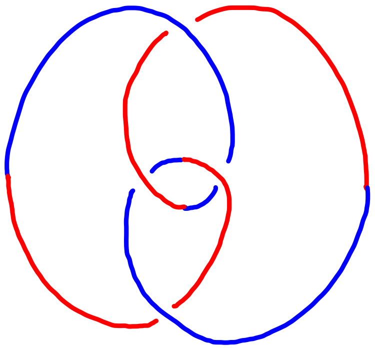


(4,7)

Ex. 4_1

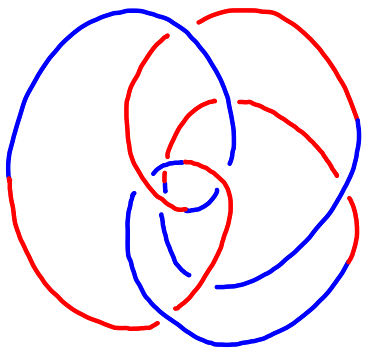
$\mathrm{g.c.d.}(n, k) = 1,\ n \geqq 3,\ k \geqq 2$

$\Rightarrow F(n, k) \in PROJ(4_1)$

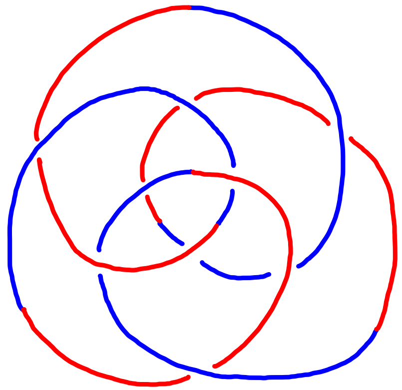


(3, 2)

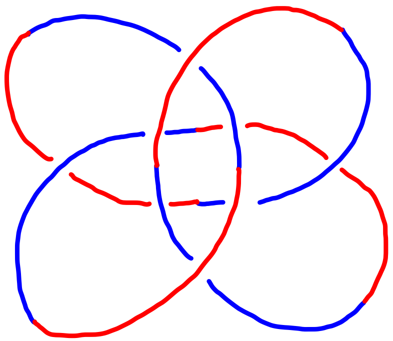
$\rightarrow$



$\approx$

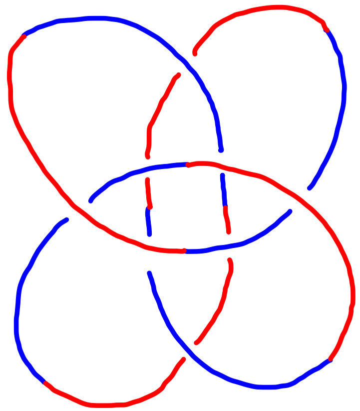


(4, 3)



(3, 4)

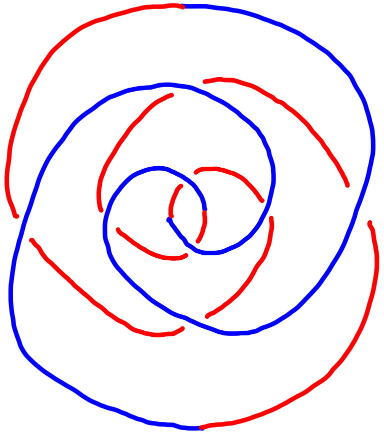
$\frac{\pi}{2}$ rotation $\rightarrow$



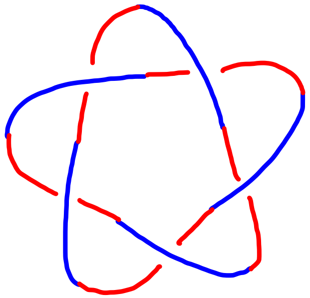
mirror image

Ex. 5_1 $\quad g.c.d.(n,k)=1,\ n,k \geqq 2,\ (n,k) \neq (2,3),(3,2)$

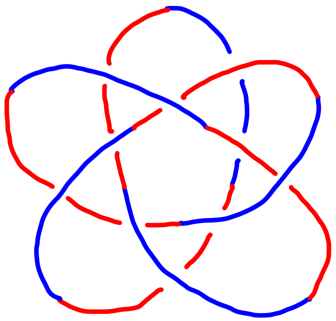
$\Rightarrow F(n,k) \in PROJ(5_1)$



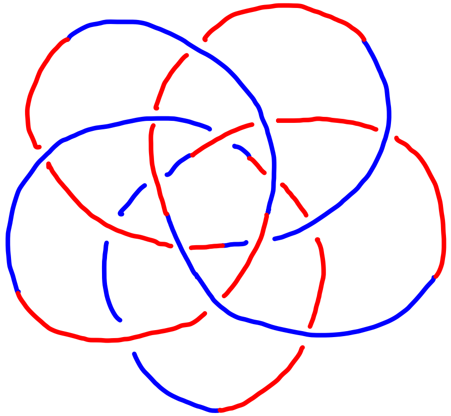
$(5,2)$



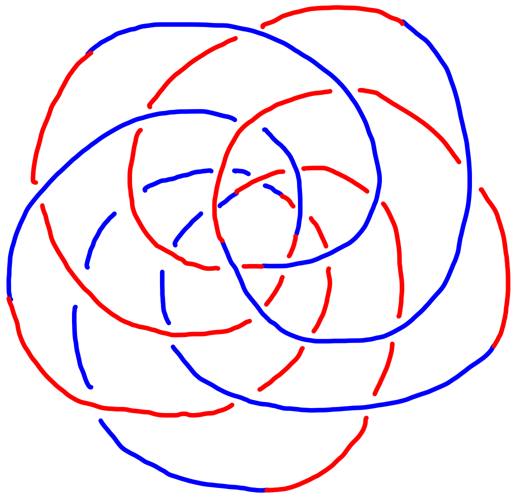
$(2,5)$



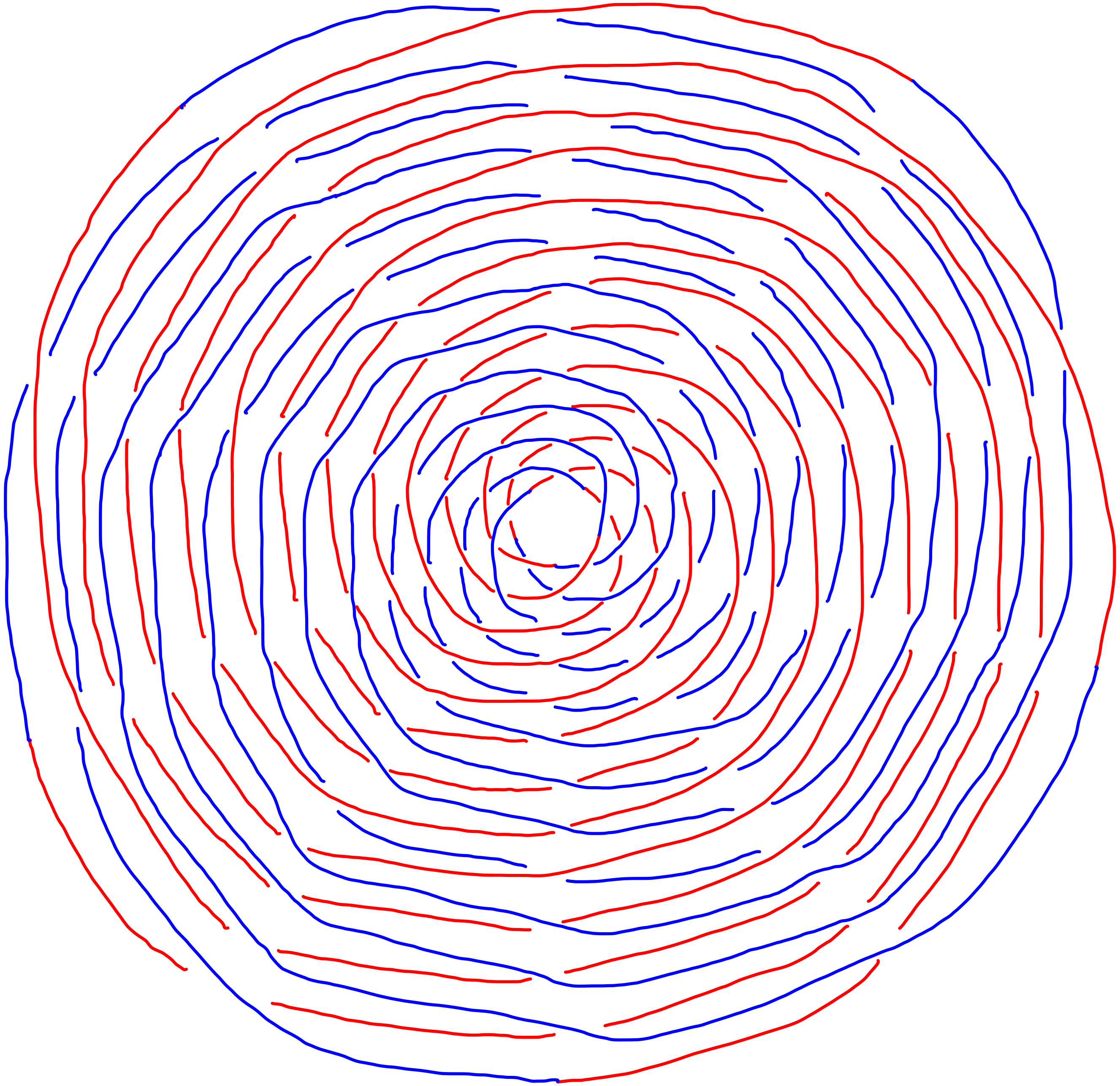
$(3,5)$



$(4,5)$

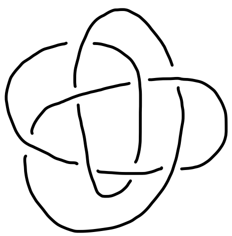


$(6,5)$

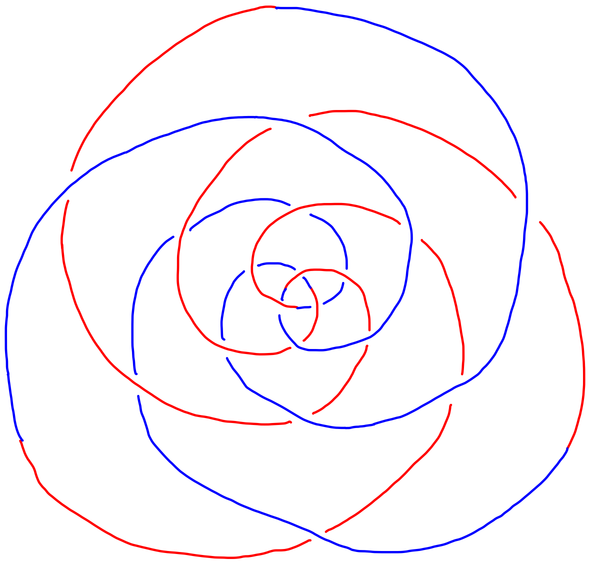


$5_1 \to F(24, 5)$

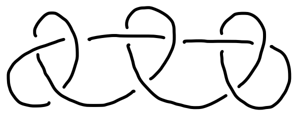
Examples



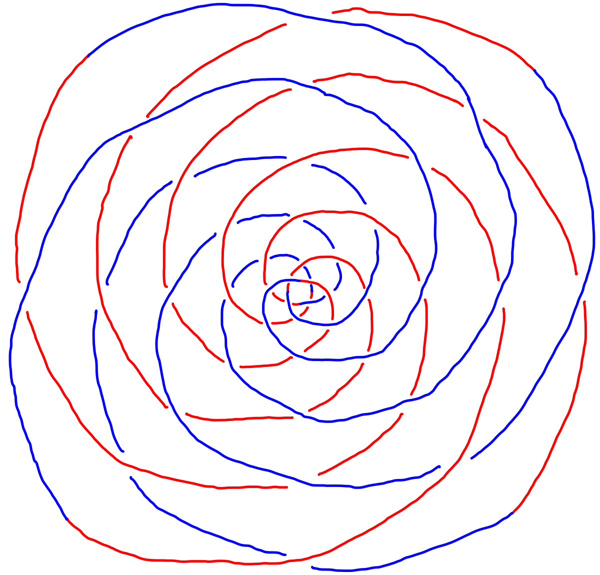
8_{17}



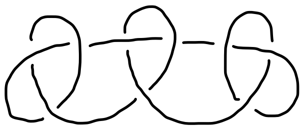
$(7, 3)$



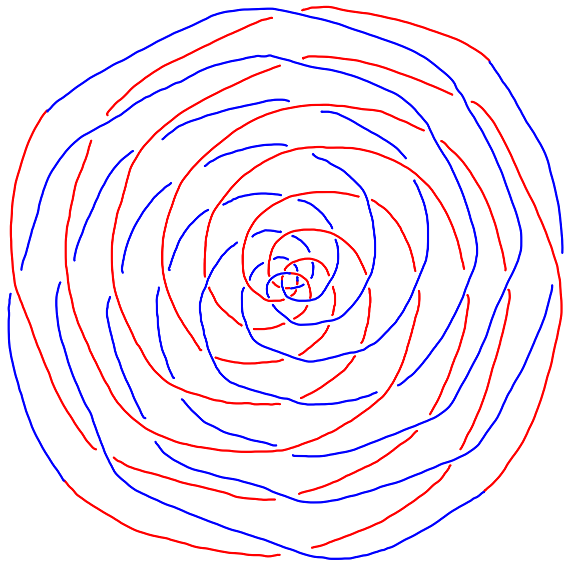
$3_1 \# 3_1 \# 3_1$



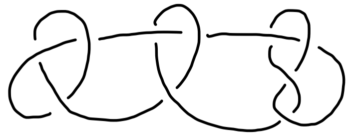
$(11, 4)$



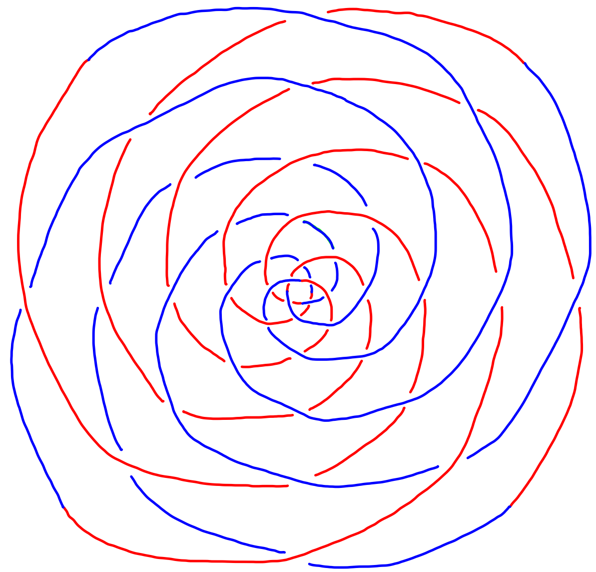
$3_1 \# 3_1 \# 3_1^*$



$(15, 4)$



$3_1 \# 3_1 \# 4_1$



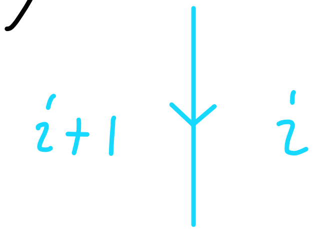
$(11, 4)$

再掲

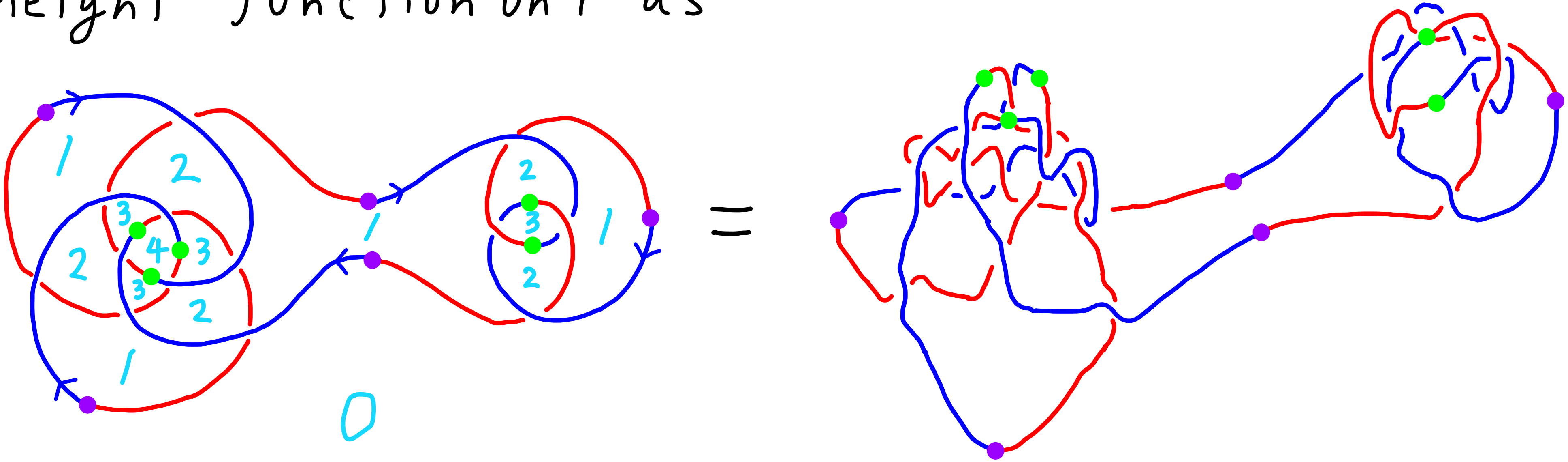
$\boxed{\underline{\text{Theorem}}\quad K : \text{knot},\ \ell r(K) = \text{bridge}(K)}$

(1) $\ell r(K) \geqq \mathrm{bridge}(K)$

$P \in PROJ(K)$, $\ell r(P) = \ell r(K)$

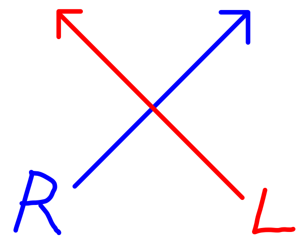
Based on Alexander numbering 

, we have a height function on P as



$L^a \longleftarrow$ descending (Alexander numbering decreasing)

$R^b \longleftarrow$ ascending (Alexander numbering increasing)

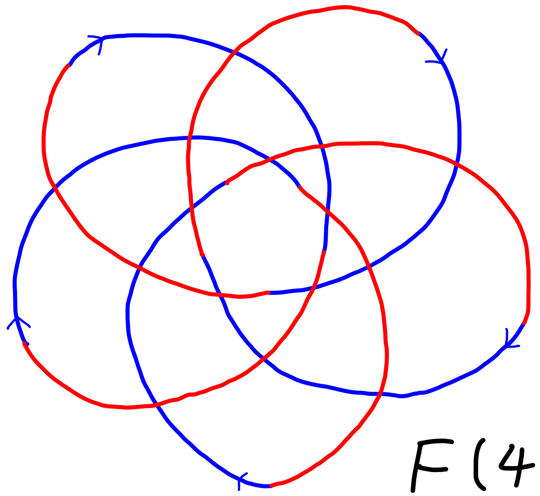


(the number of local maximums ●) = (the number of local minimums ●) = $\ell r(P)$

(2) $\ell r(K) \leqq \mathrm{bridge}(K)$

By Main theorem, $\mathrm{PROJ}(K) \ni F({}^{\exists}n, \mathrm{bridge}(K))$

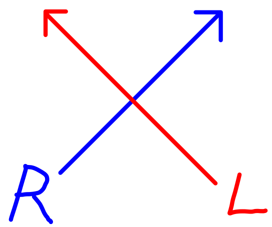
$$\boxed{\underline{\text{Prop.}} \quad \ell r(F(n,k)) = k}$$



F(4, 5)

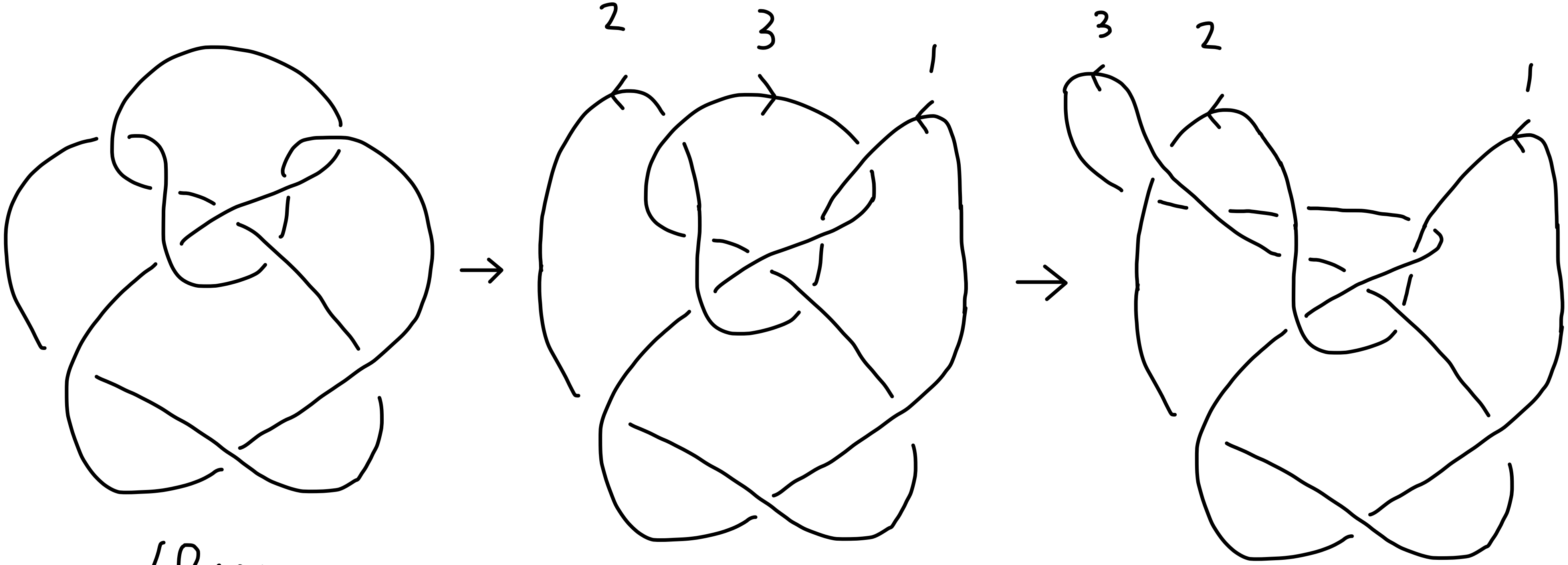
inside $\to$ outside $= L^{n-1}$

outside $\to$ inside $= R^{n-1}$



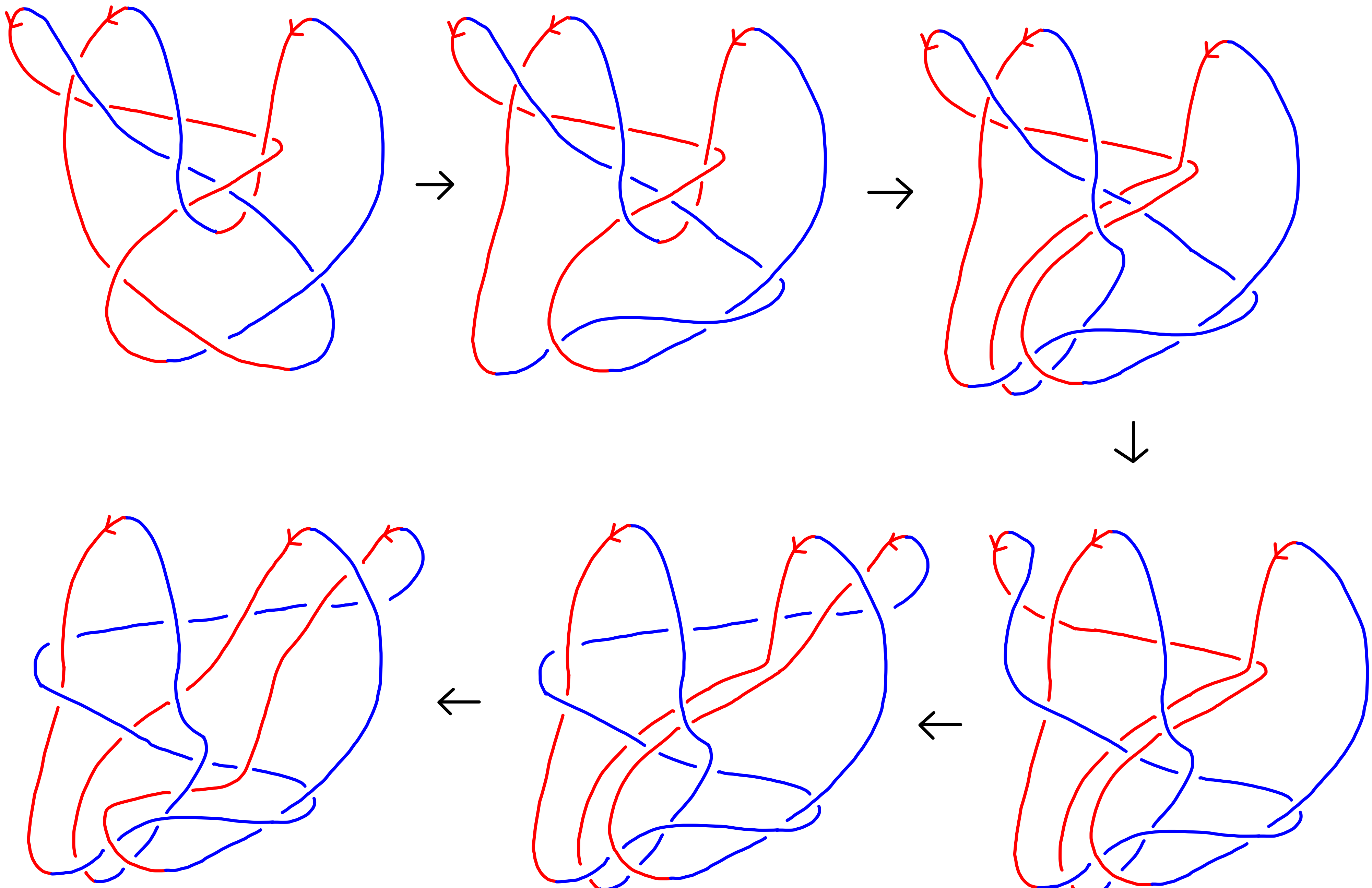
$$\therefore \ \ell r(K) \leqq \ell r\Big(F\big(n, \mathrm{bridge}(K)\big)\Big) = \mathrm{bridge}(K) \quad //$$

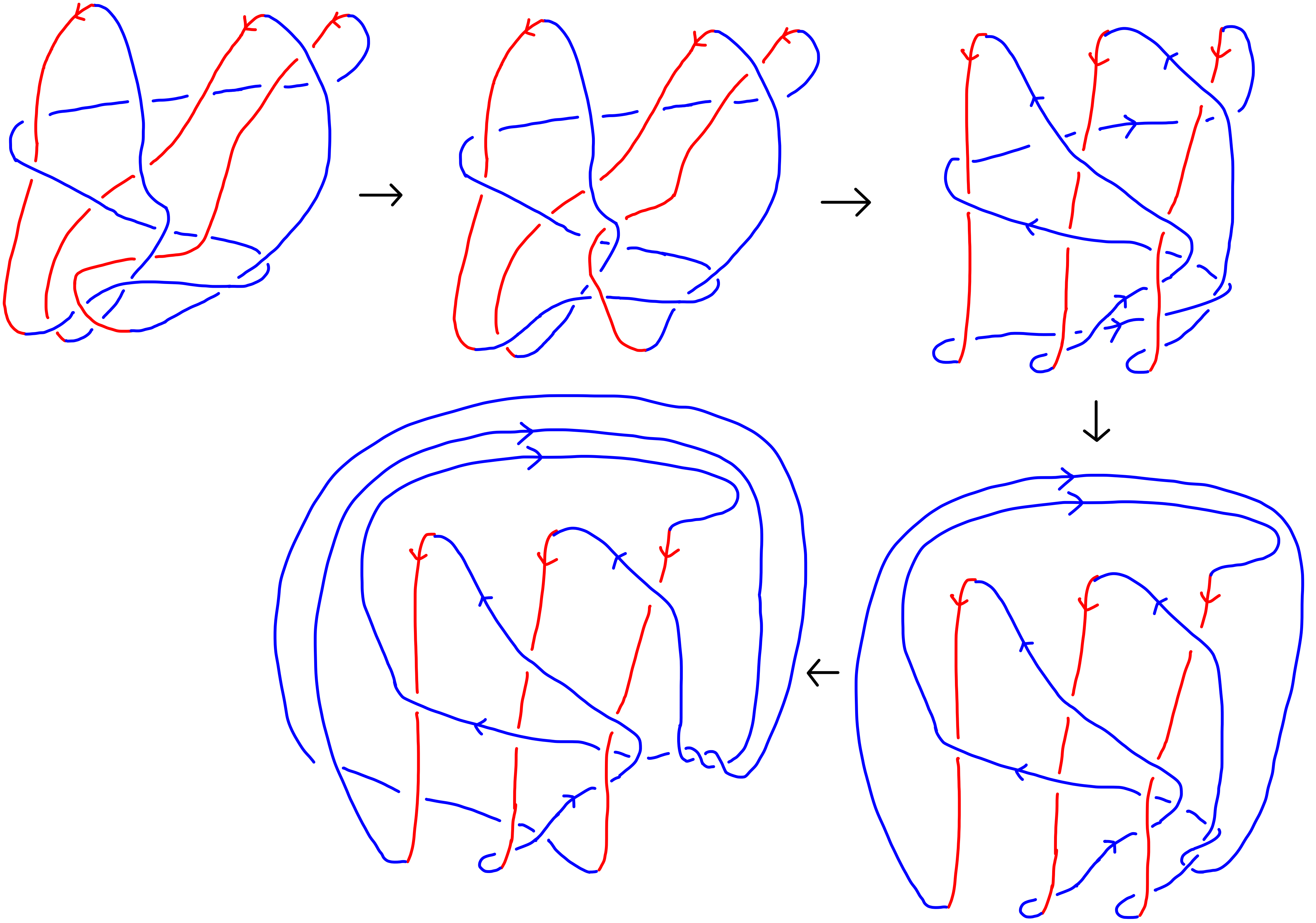
Sketch proof of Main theorem

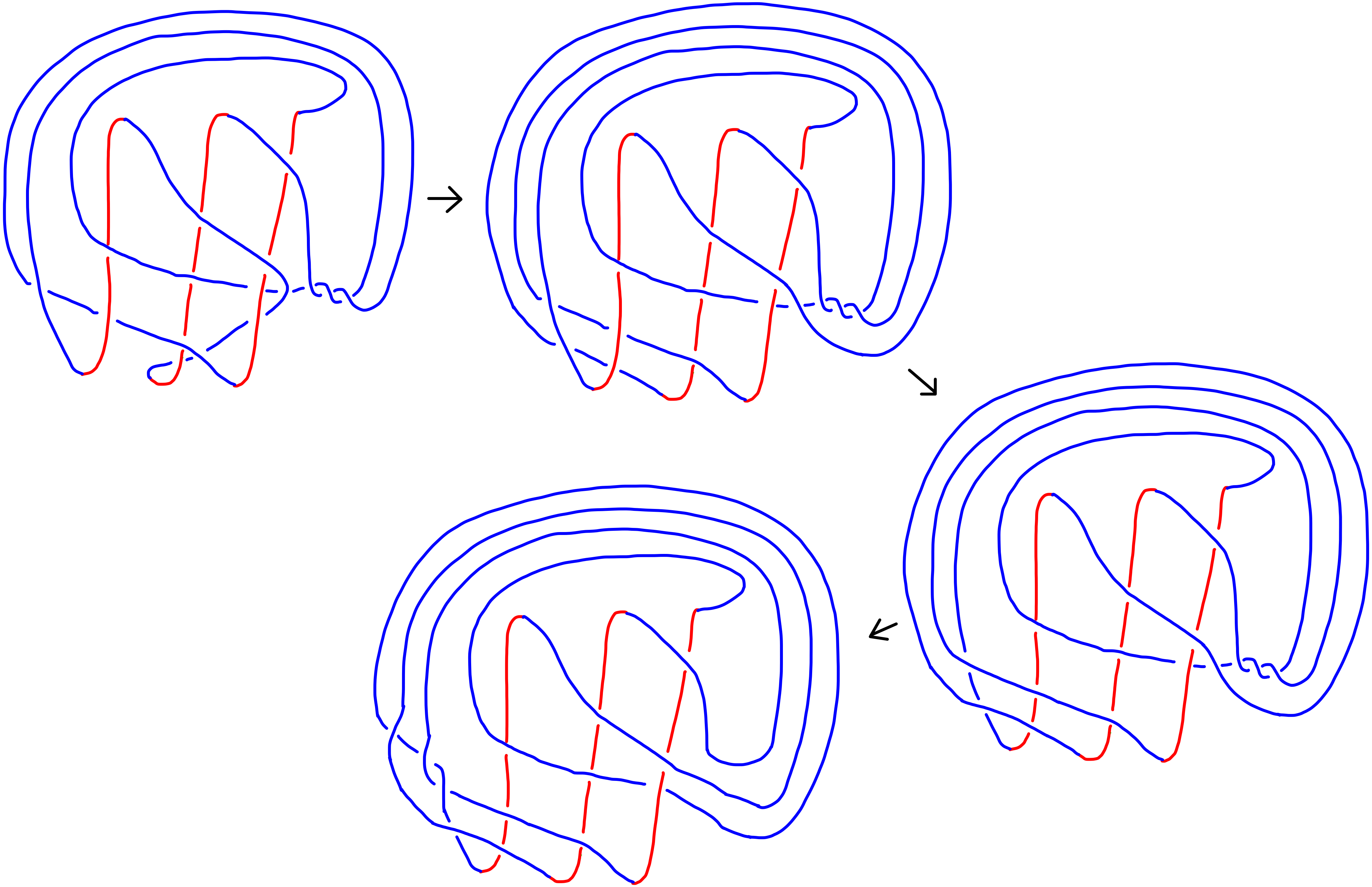


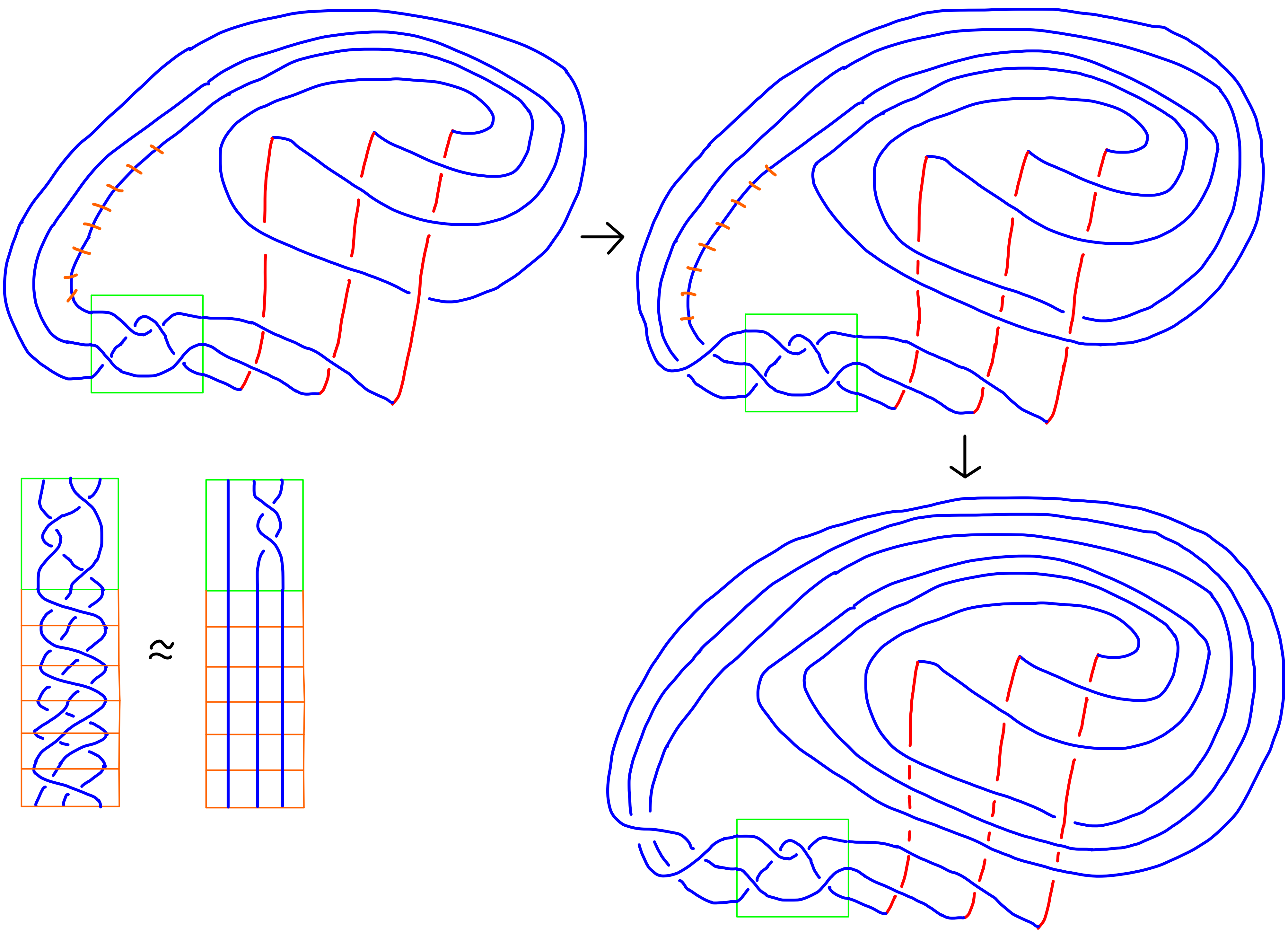
10_{161}

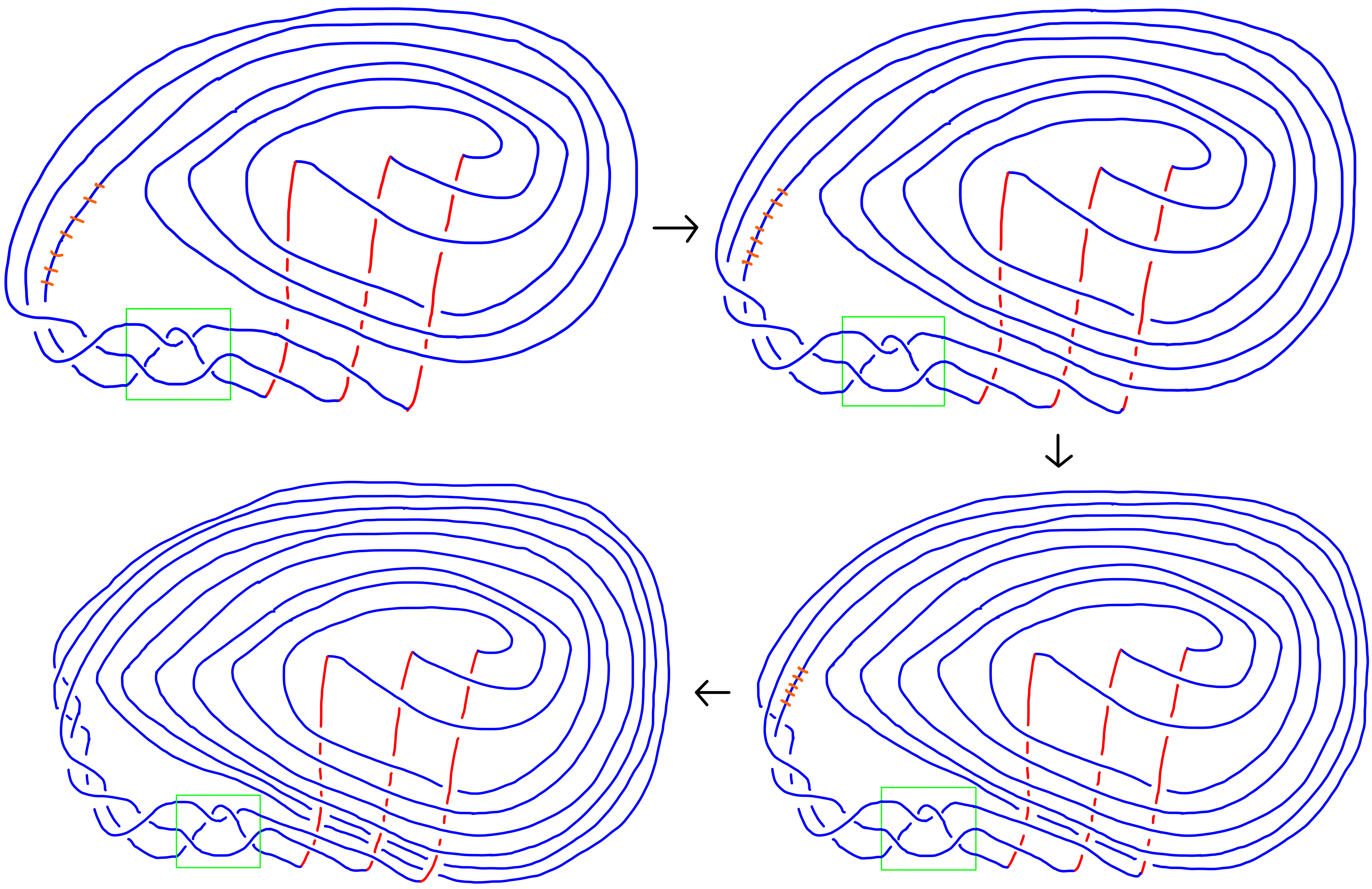
$\mathrm{bridge}(10_{161}) = 3$

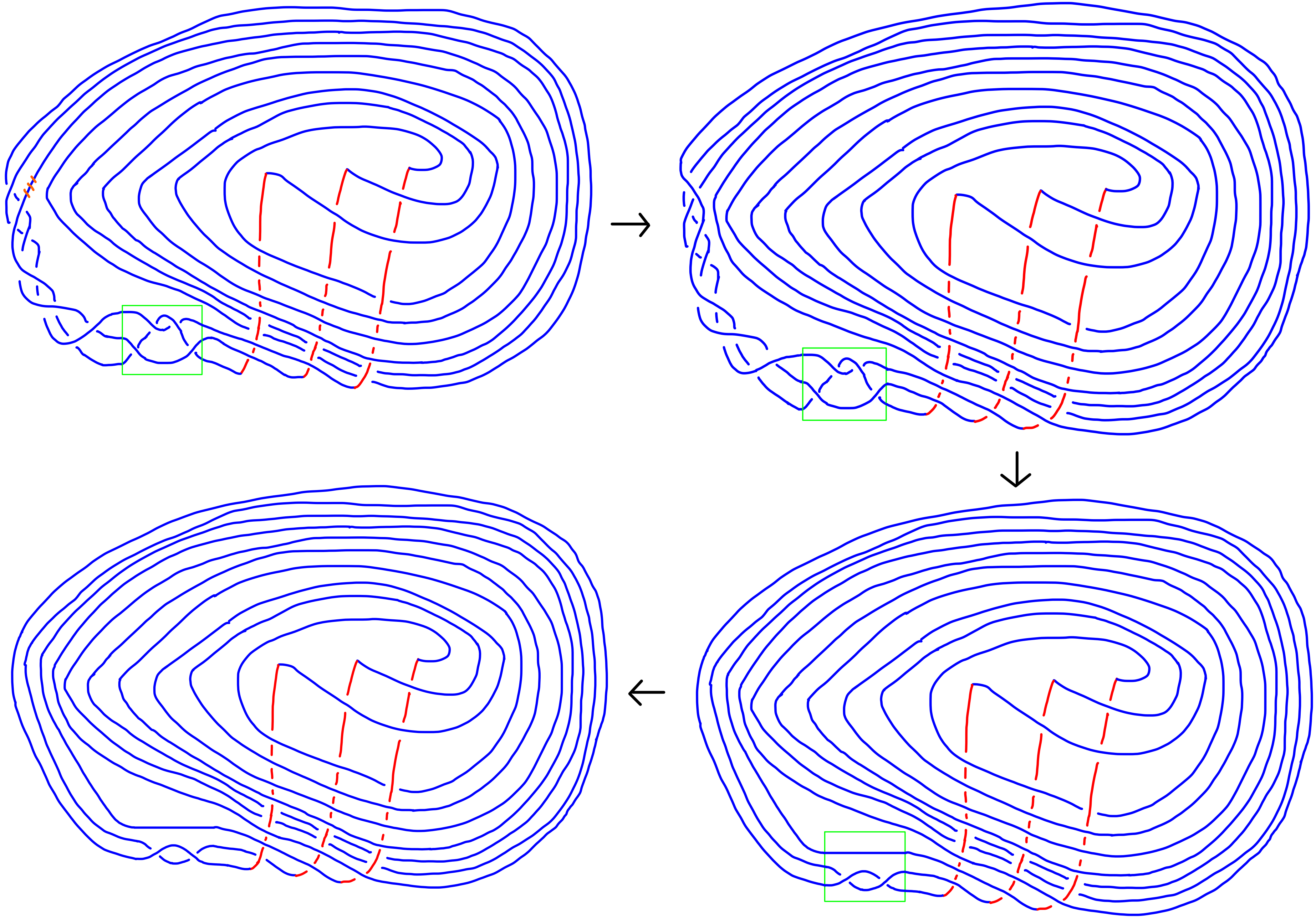


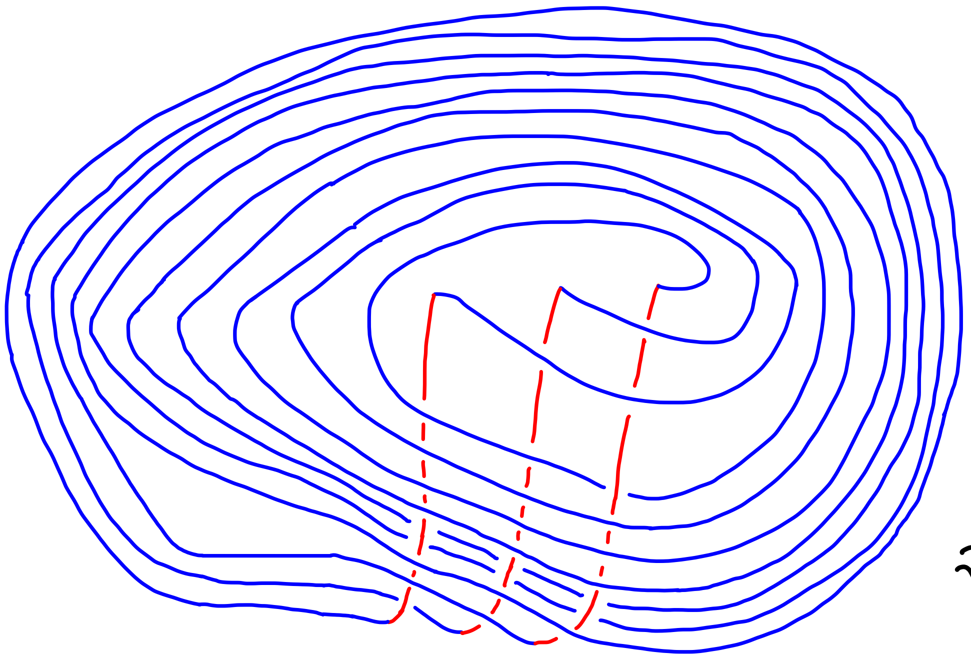




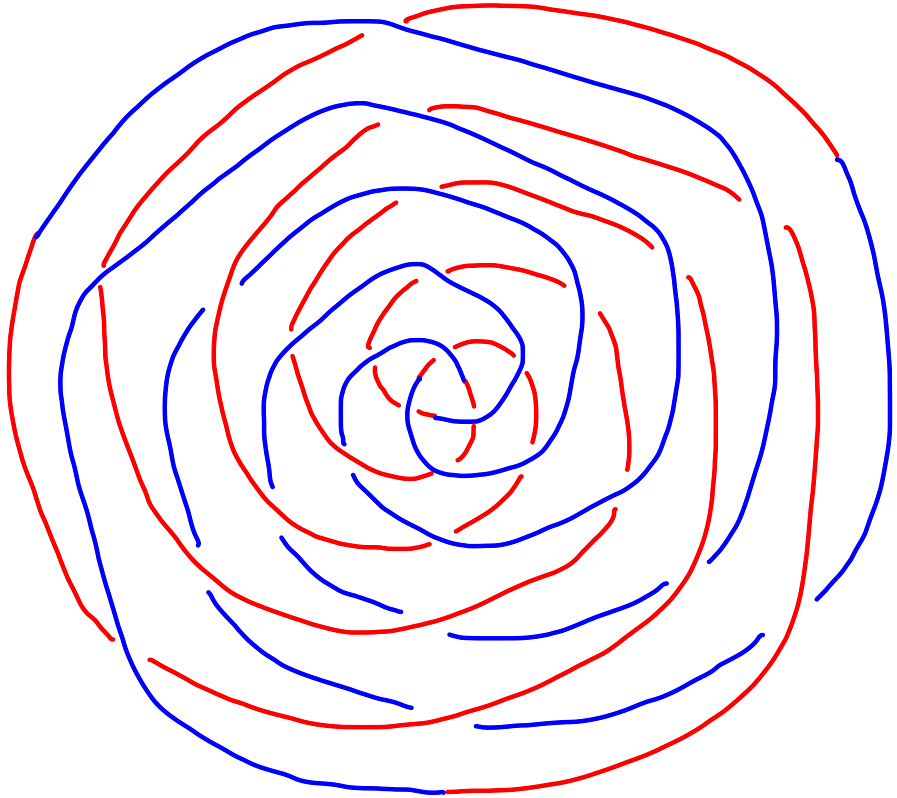








$\approx$



$F(10,3)$

//

Lemma

${}^{\forall} b$: k-braid, ${}^{\exists} n \in \mathbb{Z}_{\geq 0}$, ${}^{\exists} a_1, \cdots, a_n \in \{\alpha, \beta\}$,

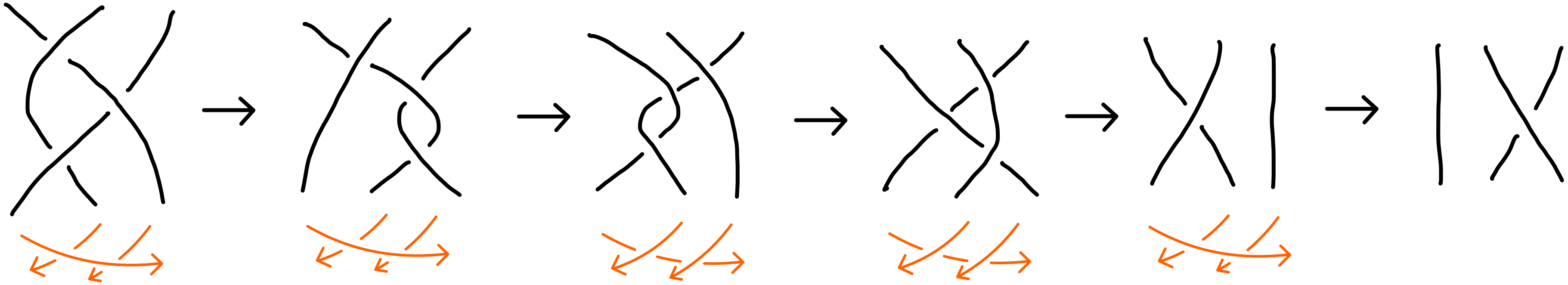
$\alpha = [\sigma_1 \sigma_2 \cdots \sigma_{k-1}]$, $\beta = [\sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{k-1}^{-1}]$

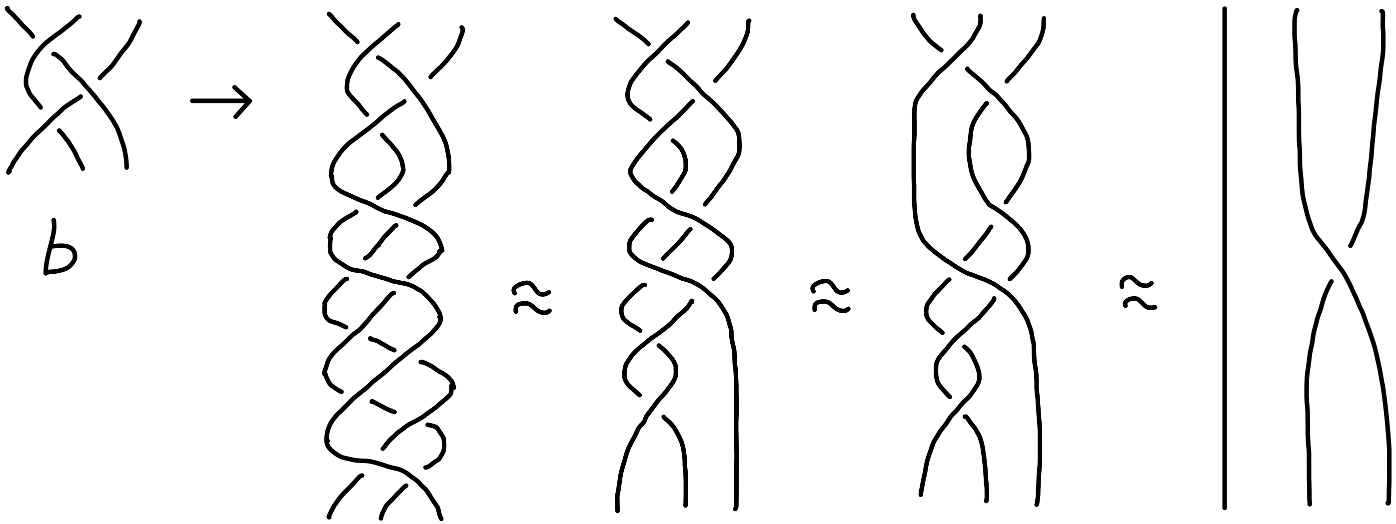
${}^{\exists} W \in \mathrm{word}(\sigma_2^{\pm 1}, \sigma_3^{\pm 1}, \cdots, \sigma_{k-1}^{\pm 1})$

s.t. $b a_1 \cdots a_n = [W]$

Example $\quad k = 3$

b





$b \qquad b\,\alpha^2\beta^2\alpha \qquad [\sigma_2]$

Thank you for your listening.

IGA BRAID (IGA KUMIHIMO)