2-irreducibility of spatial graphs

Fengchun Lei*
Department of Mathematics
Harbin Institute of Technology
Harbin 150001, P.R. China
fclei@hope.hit.edu.cn

Kouki Taniyama
Department of Mathematics
School of Education
Waseda University
Tokyo, 169-8050, Japan
taniyama@waseda.jp

Gengyu Zhang†
Department of Mathematics
Harbin Institute of Technology
Harbin 150001, P.R. China
zgy8175@yahoo.com.cn

Abstract

An embedded graph G in the 3-sphere S^3 is called 2-irreducible if there are no separating spheres, cutting spheres, singular separating spheres, singular cutting spheres and 2-cutting spheres of G. Let D be a 2-disk in S^3 that is very good for G. Let G' be an embedded graph in S^3 obtained from G by contracting D to a point. We show that if G' is 2-irreducible then G is 2-irreducible. By this criterion certain graphs are easily shown to be 2-irreducible. As an application we show a pair of embedded graphs in the 3-sphere which is distinguished by 2-irreducibility.

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†Present address: Department of Mathematics, Tokyo Institute of Technology, Tokyo, 152-8551, Japan, zhang@math.titech.ac.jp

1 Introduction

Throughout this paper we work in the piecewise linear category. Let G be a finite graph embedded in the 3-sphere S^3. We call G a spatial graph. Recall (refer to [7]) that a compact surface F embedded in S^3 is called *good for* G, if ∂F is contained in G, $\mathrm{Int}(F) \cap G$ contains at most finitely many points, and for each point $x \in \mathrm{Int}(F) \cap G$ a small neighborhood of x looks like Figure 1 for some positive integers p and q. Here x is not necessarily a vertex of G. Namely we allow the case that $p = q = 1$, and x is an interior point of an edge of G.

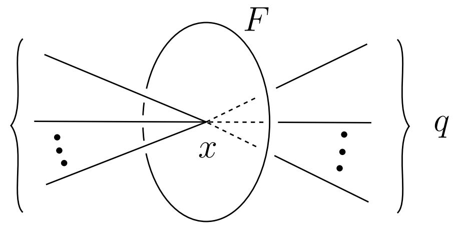

Fig. 1

Let S be a 2-sphere embedded in S^3. By $B_1(S)$ and $B_2(S)$ we denote the 3-balls in S^3 bounded by S. Namely we have $S^3 = B_1(S) \cup B_2(S)$, $B_1(S) \cap B_2(S) = \partial B_1(S) = \partial B_2(S) = S$.

A 2-sphere S embedded in S^3 is called a *separating sphere of* G if it is disjoint from G, $B_1(S) \cap G \neq \emptyset$ and $B_2(S) \cap G \neq \emptyset$. A 2-sphere S embedded in S^3 is called a *cutting sphere* of G if it is good for G and intersects G at just one point. This point is called a *cutting point*. Note that when S is a cutting sphere of G, since S is good for G, both $\mathrm{Int}(B_1(S))$ and $\mathrm{Int}(B_2(S))$ have non-empty intersection with G.

A graph G embedded in S^3 is called *irreducible* if there are no separating spheres and cutting spheres of G.

Let X be a simple arc or a 2-disk embedded in S^3. Then the quotient space S^3/X is homeomorphic to S^3. Let Y be a subspace of S^3. By Y/X we denote the image of Y under the quotient map $S^3 \to S^3/X$. We sometimes identify S^3/X with S^3 and consider Y/X as a subspace of S^3. For a finite set W we denote the number of elements of W by $|W|$.

A 2-disk D embedded in S^3 is called *fairly good for* G if it is good for G and either (a) or (b) holds:

(a) $\mathrm{Int}(D) \cap G \neq \emptyset$,

(b) $|\partial D \cap \mathrm{cl}(G - \partial D)| \neq 1$ where $\mathrm{cl}(Y)$ denotes the closure of Y.

The following is shown in [7].

Theorem 1. [7] *Let G be a graph embedded in S^3 and D a 2-disk embedded in S^3 that is fairly good for G. If the graph G/D is irreducible then G is irreducible.*

We note here that the proof of Theorem 1 is done by elementary geometric arguments. The following application of Theorem 1 is recently found.

Example 1. Let μ be a natural number greater than 2. Let $M(\mu)$ be the μ-component Milnor link defined in [1] as illustrated in Figure 2.

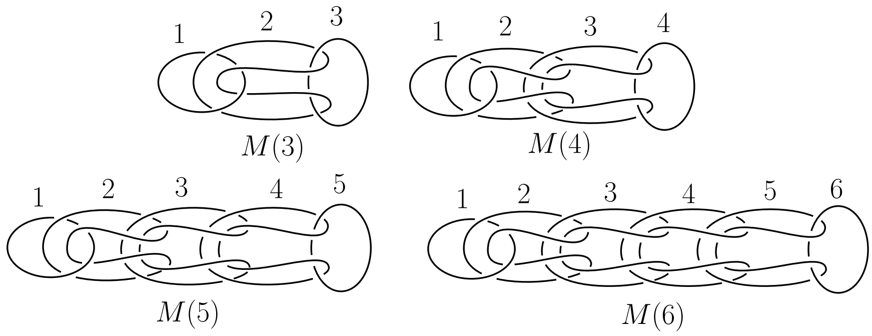

Fig. 2

By pulling apart the first two components we have that $M(\mu)$ is ambient isotopic to $M'(\mu)$ illustrated in Figure 3.

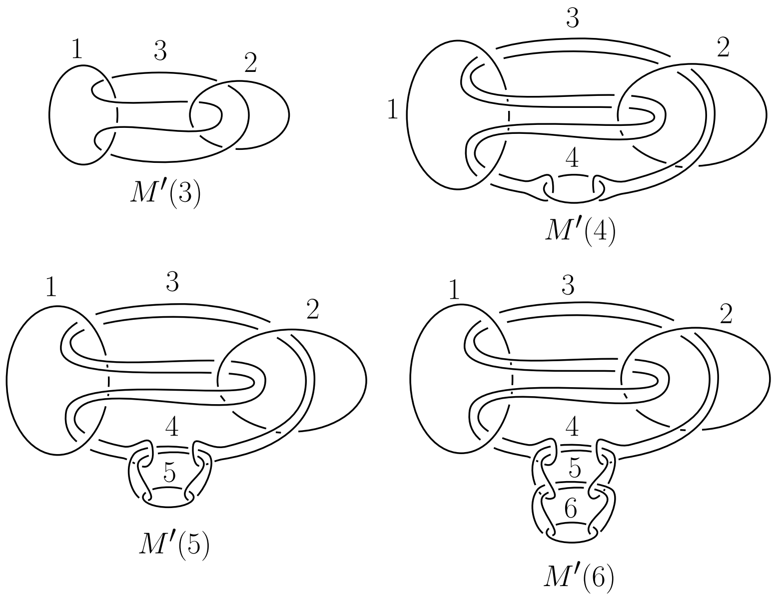

Fig. 3

Then we take fairly good disks D_1 and D_2 for $M'(\mu)$ as illustrated in Figure 4. The disk D_2/D_1 is also a fairly good disk for the graph $M'(\mu)/D_1$. Then we take fairly good disks D_3 and D_4 for $(M'(\mu)/D_1)/(D_2/D_1)$. We repeat the process and finally have a graph G' on two vertices and some edges joining them as illustrated in Figure 4.

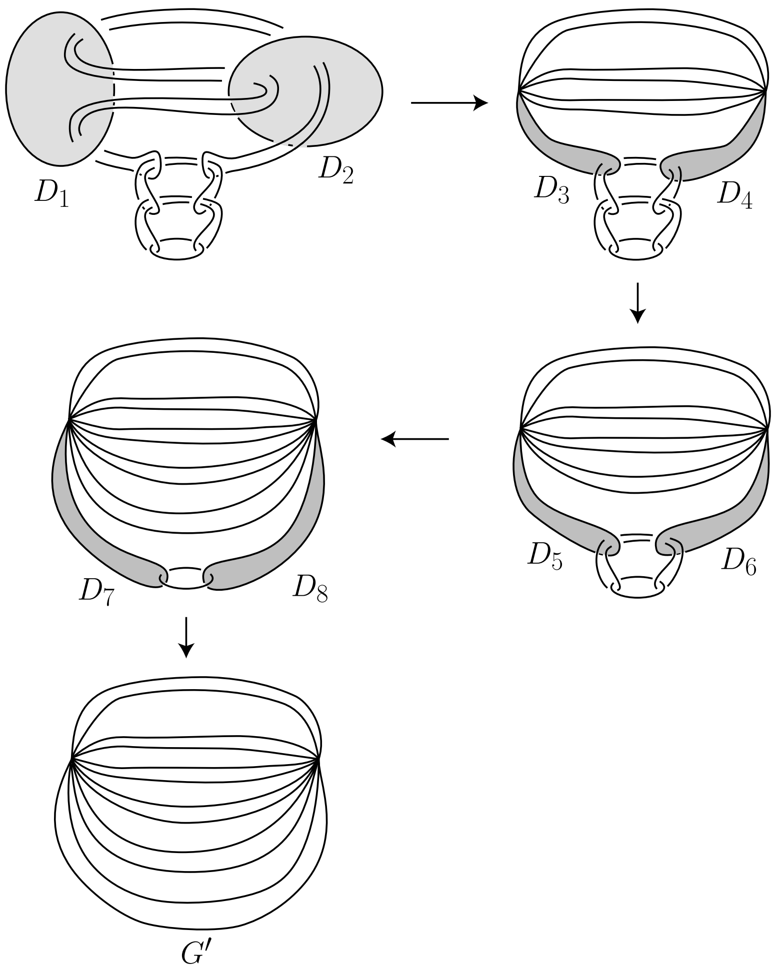

Fig. 4 ($\mu = 6$)

Since G' is connected and $G' - \{x\}$ is still connected for any point $x \in G'$ we have that G' is irreducible. Then by $2(\mu - 2)$ times applications of Theorem 1 we have that $M'(\mu)$ is, hence $M(\mu)$ is, irreducible. Therefore $M(\mu)$ is non-splittable and therefore not ambient isotopic to a μ-component trivial link $T(\mu)$. It is well-known that $M(\mu)$ and $T(\mu)$ have the same finite type invariants of order less than $\mu - 1$. This example shows that, in some cases, a simple geometric argument is as powerful as finite type invariants up to given order.

The purpose of this paper is to consider more strong decomposition of spatial graphs. Namely we define the 2-irreducibility of spatial graph and show a theorem corresponding to Theorem 1. The proof is also done by elementary geometric arguments.

Let S^2 (resp. B^3) be the unit 2-sphere (resp. unit 3-ball) centered at the origin of the 3-space. Thus $S^2 = \partial B^3$. Let $n = (1, 0, 0)$ and $s = (-1, 0, 0)$. By $S^2/(n = s)$ (resp. $B^3/(n = s)$) we denote the quotient space of S^2 (resp. B^3) under the identification $n = s$. We call a topological space that is homeomorphic to $S^2/(n = s)$ (resp. $B^3/(n = s)$) a *singular sphere* (resp. *singular 3-ball*). The identified point $n = s$ is called the *singular point*. Let S be a singular sphere embedded in S^3. Then we have that there is a singular 3-ball $B_0(S)$ in S^3 such that the frontier of $B_0(S)$ in S^3 is equal to S. Note that $B_0(S)$ may be knotted. By $s(S)$ we denote the singular point of S.

We say that a singular sphere S in S^3 is a *singular separating sphere of* G if the following conditions hold:

(a) $S \cap G = \{s(S)\}$,

(b) a small neighbourhood of $s(S)$ looks like Figure 5 for some positive integers p and q,

(c) $B_0(S) \cap G$ is not homeomorphic to a circle, and

(d) G is not contained in $B_0(S)$.

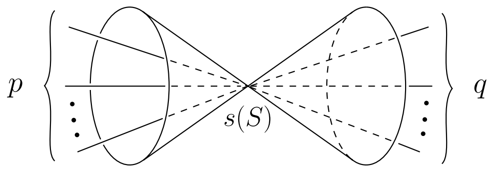

Fig. 5

We say that a singular sphere S in S^3 is a *singular cutting sphere of* G if the following conditions hold:

(a) $S \cap G = \{s(S)\}$,

(b) a small neighbourhood of $s(S)$ looks like Figure 6 for some positive integers p, q and r, and

(c) $B_0(S) \cap G$ is not homeomorphic to a circle.

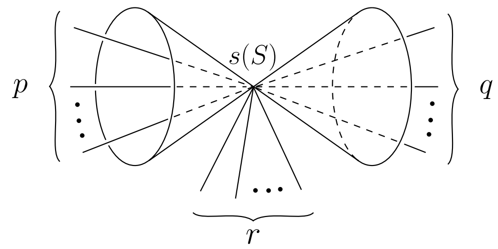

Fig. 6

We say that a 2-sphere S in S^3 is a *2-cutting sphere of* G if the following conditions hold:

(a) S is good for G and $|S \cap G| = 2$, and

(b) neither $(B_1(S) \cap G, \partial B_1(S) \cap G)$ nor $(B_2(S) \cap G, \partial B_2(S) \cap G)$ is pairwise homeomorphic to $(I, \partial I)$ where I is a closed interval.

We say that a graph G in S^3 is *2-irreducible* if there are no separating spheres, cutting spheres, singular separating spheres, singular cutting spheres, and 2-cutting spheres of G. Thus G is irreducible if G is 2-irreducible.

Let D be a 2-disk in S^3. We say that D is *very good for* G if D is good for G and one of the following conditions hold:

(a) $|\mathrm{Int}(D) \cap G| \geq 2$,

(b) $|\mathrm{Int}(D) \cap G| = 1$ and $\partial D \cap \mathrm{cl}(G - \partial D) \neq \emptyset$, or

(c) $\mathrm{Int}(D) \cap G = \emptyset$ and $|\partial D \cap \mathrm{cl}(G - \partial D)| \neq 1, 2$.

Thus D is fairly good for G if D is very good for G.

The following is the main theorem of this paper.

Theorem 2. *Let G be a graph embedded in S^3 and D a 2-disk embedded in S^3 that is very good for G. If the graph G/D is 2-irreducible then G is 2-irreducible.*

Example 2. Let G_1, G_2 and G_3 be spatial graphs as illustrated in Figure 7. Since G_1 has a separating sphere G_1 is not irreducible. By contracting the six obvious very good disks for G_2 and G_3 respectively we have the spatial graphs G_4 and G_5 as illustrated in Figure 7. Since G_4 and G_5 are irreducible we have by six times applications of Theorem 1 that both G_2 and G_3 are irreducible. Thus we have that these two are not ambient isotopic to G_1. Since G_2 has a 2-cutting sphere G_2 is not 2-irreducible. Note that G_5 is connected and $G - \{x\}$ is still connected for any $x \in G_5$, and if $G_5 - \{x, y\}$ is not connected for some $x, y \in G_5$ then $G_5 - \{x, y\}$ has just two components and one of the closures of them is homeomorphic to a closed interval. Hence we have that G_5 is 2-irreducible. Then by six times applications of Theorem 2 we have that G_3 is 2-irreducible. Thus G_2 and G_3 are not distinguished by the irreducibility but distinguished by the 2-irreducibility. Note that both G_2 and G_3 are minimally knotted, namely both of them are nontrivial but trivial if any one of the edges are removed.

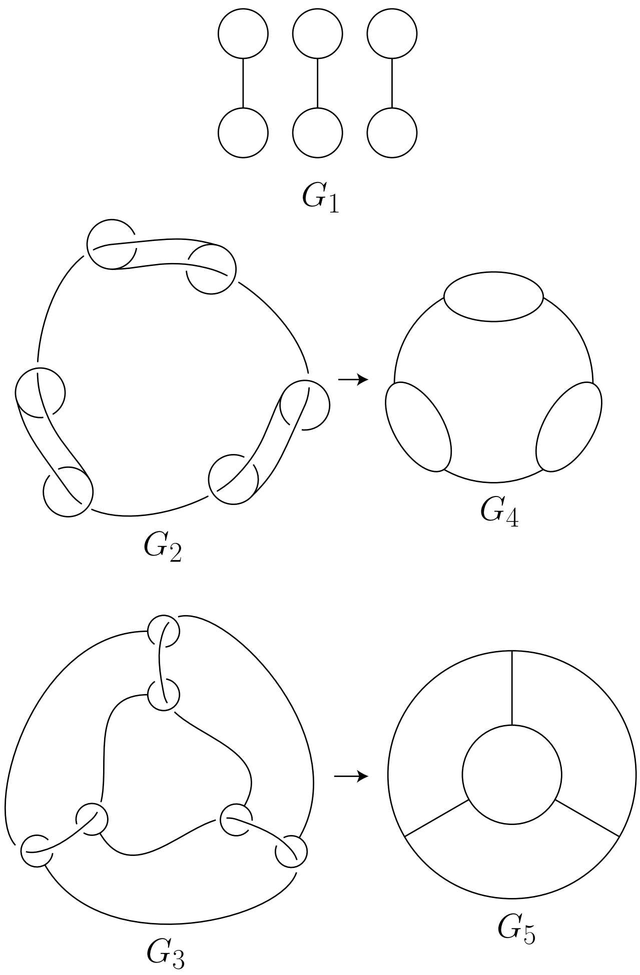

Fig. 7

Remark. There are some preceding works related to decompositions of spatial graphs by spheres embedded in S^3. We refer, among others, to [6], [4], [5], [2] and [3].

2 Proof

Lemma 1. *Let G be a graph embedded in S^3 and D a 2-disk embedded in S^3 that is good for G. Let S be a cutting sphere of G such that $B_1(S) \cap G \neq \partial D$ and $B_2(S) \cap G \neq \partial D$. Then G/D has a separating sphere or a cutting sphere.*

Proof. If D is fairly good for G then we have the result by Theorem 1. Suppose that D is not fairly good. Then we have by definition that $\mathrm{Int}(D) \cap G = \emptyset$ and $|\partial D \cap \mathrm{cl}(G - \partial D)| = 1$. In this case we may identify G/D with $\mathrm{cl}(G - \partial D)$. Then by the assumption we have that S is still a cutting sphere of $\mathrm{cl}(G - \partial D)$, or, after a slight push off near the cutting point, S becomes a separating sphere of $\mathrm{cl}(G - \partial D)$. Thus we have the conclusion. □

Proof of Theorem 2. It is sufficient to show that if G has a separating sphere, a cutting sphere, a singular separating sphere, a singular cutting sphere or a 2-cutting sphere then G/D also has one of them. The case that G has a separating sphere or a cutting sphere is shown by Theorem 1. Therefore we consider the case that G has a singular separating sphere, a singular cutting sphere or a 2-cutting sphere. We may suppose that S and D are in general position. Then we have that $S \cap D$ consists of 'circles', 'loops', 'arcs' and 'points'. Here by a 'point' we mean a point in $S \cap G \cap D$. Therefore there are at most two 'points' of $S \cap D$. By a 'circle' we mean a component of $S \cap D$ that is homeomorphic to a circle, and is disjoint from the set of 'points'. By a 'loop' we mean a subspace of $S \cap D$ that is homeomorphic to a circle, and contains just a 'point'. By an 'arc' we mean a subspace of $S \cap D$ that is homeomorphic to a closed interval, and the end points are the two 'points'. Let S be a singular separating sphere, a singular cutting sphere or a 2-cutting sphere of G that is in general position with respect to D such that the total number of 'circles', 'loops' and 'arcs' is minimal among all such spheres of G. We show the following claim by an induction on the total number of 'circles', 'loops' and 'arcs'.

Claim. *Let G be a graph embedded in S^3 and D a 2-disk embedded in S^3 that is very good for G. Let S be a singular separating sphere, a singular cutting sphere or a 2-cutting sphere of G that is in general position with respect to D. Then G/D has a separating sphere, a cutting sphere, a singular separating sphere, a singular cutting sphere or a 2-cutting sphere.*

First we consider the case that $S \cap D$ contains no 'circles', 'loops' and 'arcs'. Note that $S \cap D = S \cap D \cap G$ and $|S \cap D| = 0, 1,$ or 2.

Case 1. $|S \cap D| = 0$.

Since D is very good for G we have that the point D/D is a vertex of the graph G/D of degree 0 or of degree greater than two. Therefore we have that S $(= S/D)$ is still a singular separating sphere, a singular cutting sphere or a 2-cutting sphere of G/D.

Case 2. $|S \cap D| = 1$.

Let x be the point in $S \cap D$ so that $S \cap D = \{x\}$. There are two cases that $x \in \partial D$ and $x \in \text{Int}(D)$. In any case we have, by the very goodness of D, that S $(= S/D)$ is still a singular separating sphere, a singular cutting sphere or a 2-cutting sphere of G/D.

Case 3. $|S \cap D| = 2$.

Let x and y be the points in $S \cap D$ so that $S \cap D = \{x, y\}$. In this case S is a 2-cutting sphere. There are three cases that both of x, y, one of x, y, none of x, y is contained in ∂D. In any case we have that S/D is a singular cutting sphere of G/D.

Next we consider the case that $S \cap D$ contains some 'circles', 'loops' or 'arcs'. First suppose that there is an innermost circle or loop on S such that the interior of the innermost disk on S does not intersect with G. Let δ be the innermost disk on S. Let σ be the disk on D with $\partial\sigma = \partial\delta$. If $\text{Int}(\sigma) \cap G \neq \emptyset$ then we have that δ/D is a cutting sphere of G/D. Suppose that $\text{Int}(\sigma) \cap G = \emptyset$. Let $B_1(\delta \cup \sigma)$ be the 3-ball bounded by $\delta \cup \sigma$ with $\text{Int}(B_1(\delta \cup \sigma)) \cap \partial D = \emptyset$. Suppose that $\text{Int}(B_1(\delta \cup \sigma)) \cap G \neq \emptyset$. Then we have that δ/D is a cutting sphere of G/D, or, after a slight push off near the point $\partial\delta/D$, δ/D becomes a separating sphere of G/D. Suppose that $\text{Int}(B_1(\delta \cup \sigma)) \cap G = \emptyset$. Then by an ambient isotopy along $B_1(\delta \cup \sigma)$ we deform δ to σ and have a new S with fewer intersection with D.

Therefore it is sufficient to consider the case that every innermost circle or loop bounds an innermost disk on S which contains just a point of $S \cap G$ in its interior. There are the following four cases.

Case A. S is a 2-cutting sphere and $(S \cap D) - (S \cap G)$ is a disjoint union of some parallel 'circles' each of which separates the two points in $S \cap G$ on S.

Case B. S is a singular separating sphere or a singular cutting sphere and $(S \cap D) - s(S)$ is a disjoint union of some parallel 'circles' each of which does not separate S.

Case C. S is a singular separating sphere or a singular cutting sphere and $S \cap D$ is a union of some 'loops' each of which does not separate S.

Case D. S is a 2-cutting sphere and $S \cap D$ is a union of some 'arcs'.

First we consider Case A and Case B. Let σ be an innermost disk on D. There are six cases as follows.

Case A1. $S \cap \sigma = \partial\sigma$.

Case A2. $S \cap \sigma = \partial\sigma \cup \{x\}$ where x is a point in $S \cap G$.

Case A3. $S \cap \sigma = \partial\sigma \cup (S \cap G)$.

Case B1. $S \cap \sigma = \partial\sigma$.

Case B2. $S \cap \sigma = \partial\sigma \cup \{s(S)\}$ and $\sigma \subset B_0(S)$.

Case B3. $S \cap \sigma = \partial\sigma \cup \{s(S)\}$ and σ is not contained in $B_0(S)$.

Note that in Case B1 we have $\sigma \subset B_0(S)$. Suppose that $\mathrm{Int}(\sigma) \cap G = \emptyset$. Note that this happens in Case A1 and Case B1. Then we obtain a cutting sphere of G by cutting S along σ.

Next suppose that $|\mathrm{Int}(\sigma) \cap G| = 1$. Note that this happens in Case A1, Case A2, Case B1, Case B2 and Case B3. In Case A1 and Case B1 we have at least one 2-cutting sphere of G by cutting S along σ. In Case A2 we cut S along σ and have a 2-sphere S_1 and a singular sphere S_2. Let $B_1(S)$ be the 3-ball bounded by S with $B_1(S) \supset \sigma$ and let $B_1(S_1)$ be the 3-ball bounded by S_1 with $B_1(S_1) \subset B_1(S)$. Since $(B_2(S) \cap G, \partial B_2(S) \cap G)$ is not pairwise homeomorphic to $(I, \partial I)$ we have that $(B_2(S_1) \cap G, \partial B_2(S_1) \cap G)$ is not pairwise homeomorphic to $(I, \partial I)$. If $(B_1(S_1) \cap G, \partial B_1(S_1) \cap G)$ is not pairwise homeomorphic to $(I, \partial I)$ then we have that S_2 is a 2-cutting sphere of G and, after a slight deformation, has fewer intersection with D. Suppose that $(B_1(S_1) \cap G, \partial B_1(S_1) \cap G)$ is pairwise homeomorphic to $(I, \partial I)$. Since $(B_1(S) \cap G, \partial B_1(S) \cap G)$ is not pairwise homeomorphic to $(I, \partial I)$ we have that $\mathrm{Int}(B_1(S) - B_1(S_1)) \cap G \neq \emptyset$. Then we have that S_2 is a singular separating sphere of G or a singular cutting sphere of G. After a slight deformation S_2 has fewer intersection with D. In Case B2 and in Case B3 by similar arguments we have singular separating sphere or a singular cutting sphere of G with fewer intersection with D by cutting S along σ.

Next suppose that $|\mathrm{Int}(\sigma) \cap G| \geq 2$. Let S_1 and S_2 be two 2-spheres, a 2-sphere and a singular sphere, or two singular spheres such that $S/\sigma = S_1 \cup S_2$. Then we have that each of them is a 2-cutting sphere or a singular cutting sphere of G/σ. If the 2-disk D/σ is very good for G/σ then we have the claim by the hypothesis of induction. Suppose that D/σ is not very good for G/σ. Then we have that $|\mathrm{Int}(D/\sigma) \cap G/\sigma| = 1$ and $\partial(D/\sigma) \cap \mathrm{cl}(G/\sigma - \partial(D/\sigma)) = \emptyset$. Then we may identify $G/D = (G/\sigma)/(D/\sigma)$ with $G/\sigma - \partial(D/\sigma)$. Then we have that each of S_1 and S_2 is a 2-cutting sphere of $G/\sigma - \partial(D/\sigma)$ or a singular cutting sphere of $G/\sigma - \partial(D/\sigma)$.

Next we consider Case C. Let σ be an innermost disk on D. It is clear that σ is contained in $\mathrm{cl}(S^3 - B_0(S))$. Suppose that $\mathrm{Int}(\sigma) \cap G = \emptyset$. Then by cutting S along σ we have a cutting sphere of G, or, after a slight deformation near $s(S)$, we have a separating sphere of G. Suppose that $\mathrm{Int}(\sigma) \cap G \neq \emptyset$. Then we have that S/σ is a cutting sphere of G/σ and the disk D/σ is good for G/σ. By the condition that $B_0(S) \cap G$ is not homeomorphic to a circle and by the condition $\mathrm{Int}(\sigma) \cap G \neq \emptyset$ we have that $B_1(S/\sigma) \cap (G/\sigma) \neq \partial(D/\sigma)$ and $B_2(S/\sigma) \cap (G/\sigma) \neq \partial(D/\sigma)$. Then by Lemma 1 we have the conclusion.

Next we consider Case D. There are three cases as follows.

Case D1. The two points in $S \cap G$ are contained in $\mathrm{Int}(D)$.

Case D2. Just one of the two points in $S \cap G$ is contained in $\mathrm{Int}(D)$.

Case D3. The two points in $S \cap G$ are contained in ∂D.

First we consider Case D1 and Case D2. Let α be one of the 'arcs'. Then we have that S/α is a cutting sphere of G/α. In Case D1 the disk D/α is fairly good for G/α and we have the result by Theorem 1. In Case D2 it holds that $B_1(S/\alpha) \cap (G/\alpha) \neq \partial(D/\alpha)$ and $B_2(S/\alpha) \cap (G/\alpha) \neq \partial(D/\alpha)$. Then by Lemma 1 we have the result.

Next we consider Case D3. Since D is very good for G we have that there is a point in $\mathrm{Int}(D) \cap G$, or there is a point in $\partial D \cap \mathrm{cl}(G - \partial D)$ that is not a point in $G \cap S$. Let x be such a point. Let α be an 'arc' that is outermost on D. Namely there are two disks D_1 and D_2 such that $D = D_1 \cup D_2$, $D_1 \cap D_2 = \alpha$ and $D_2 \cap S = \alpha$. By choosing another outermost 'arc' if necessary we may suppose that x is contained in D_1. Note that S/α is a cutting sphere of G/α. By the condition that neither $(B_1(S) \cap G, \partial B_1(S) \cap G)$ nor $(B_2(S) \cap G, \partial B_2(S) \cap G)$ is pairwise homeomorphic to $(I, \partial I)$ we have that $(S/\alpha)/(D_2/\alpha)$ is still a cutting sphere of $(G/\alpha)/(D_2/\alpha)$, or, after a slight push off near the cutting point, $(S/\alpha)/(D_2/\alpha)$ becomes a separating sphere of $(G/\alpha)/(D_2/\alpha)$. In case that $x \in \mathrm{Int}(D) \cap G$ we have that $(D_1/\alpha)/(D_2/\alpha)$ is a fairly good disk for $(G/\alpha)/(D_2/\alpha)$ and we have the result by Theorem 1. Now we consider the case that $x \in \partial D \cap \mathrm{cl}(G - \partial D)$. If $(S/\alpha)/(D_2/\alpha)$ is still a cutting sphere of $(G/\alpha)/(D_2/\alpha)$ then we have that $(D_1/\alpha)/(D_2/\alpha)$ is a fairly good disk for $(G/\alpha)/(D_2/\alpha)$ and we have the result by Theorem 1. If $(S/\alpha)/(D_2/\alpha)$ becomes a separating sphere of $(G/\alpha)/(D_2/\alpha)$ after a slight push off near the cutting point, then $(D_1/\alpha)/(D_2/\alpha)$ may not be a fairly good disk for $(G/\alpha)/(D_2/\alpha)$. But then we may identify $G/D = ((G/\alpha)/(D_2/\alpha))/((D_1/\alpha)/(D_2/\alpha))$ with $\mathrm{cl}(((G/\alpha)/(D_2/\alpha)) - \partial((D_1/\alpha)/(D_2/\alpha)))$, and we have that $(S/\alpha)/(D_2/\alpha)$ is a separating sphere of $\mathrm{cl}(((G/\alpha)/(D_2/\alpha)) - \partial((D_1/\alpha)/(D_2/\alpha)))$. This completes the proof of Claim and we have proved Theorem 2. □

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